Lecture 4-21

Continuing from where we left off last time, we would like to see now that the sign of a permutation of n indices is well defined; that is, if a permutation is a product of oddly many interchanges of adjacent indices, then it cannot also be the product of evenly many such interchanges. To do this we attach a nonnegative integer $\ell(\pi)$ to a permutation π in such a way that this number changes by one if π is multiplied on the left by an interchange of adjacent indices. This number $\ell(\pi)$, called the *length* of π (or the *disorder* of π in the Treil notes) is the number of pairs (i, j) of indices such that $i < j$ but $\pi(i) > \pi(j)$; equivalently, if the numbers $\pi(1), \ldots, \pi(n)$ are written in this order along a line, it is the number of times a larger number precedes a smaller one. Now it is easy to check that, given the arrangement $\pi(1), \ldots, \pi(n)$, if two adjacent numbers in it are interchanged, then the length increases or decreases by one; it increases by one if the leftmost of the numbers interchanged is less than its right-hand neighbor and decreases by one otherwise. Since the length of the identity permutation is 0 (and in fact the identity is the only permutation of length 0), it follows that the product of an odd number of adjacent interchanges (or any interchanges) cannot also be the product of an even number of adjacent interchanges (or any interchanges), as claimed.

Thus there is indeed one and only one multilinear alternating function f of the rows $A_1, \ldots, A_n$ of an $n \times n$ matrix A such that $f(\vec{e}_1, \ldots, \vec{e}_n) = 1$. Writing as before a_{ij} for the ij-th entry of A, we have $f(A_1, \ldots, A_n) = \det A = \sum_\pi s(\pi) a_{1\pi(1)} \ldots a_{n\pi(n)}$, where $s(\pi) = \mathrm{sgn}(\pi)$ is the sign of the permutation π. This sum has only $n!$ terms rather than the n^n terms of the first sum we wrote down last time, but that is still a lot of terms. Fortunately, there is a much easier inductive formula for the determinant of an $n \times n$ matrix, expressing this determinant in terms of determinants of various $(n-1) \times (n-1)$ matrices. To write down this formula we need some more notation. For any indices i, j between 1 and n, write A^{ij} for the *ij-minor* of A, obtained from A by deleting its i-th row and j-th column. Write C^{ij} for the *ij-cofactor* of A, equal to $(-1)^{i+j}$ times the determinant $\det A^{ij}$. Then for each fixed i we have $\det A = \sum_{j=1}^n a_{ij} C^{ij}$; likewise for each fixed j we have $\det A = \sum_{i=1}^n a_{ij} C^{ij}$. We call the first of these formulas the *expansion of* $\det A$ *about the i-th row of A*; the second one is similarly the *expansion of* $\det A$ *about the j-th column of A*. These formulas hold because, given a permutation π of the n indices, it takes $\pi(1) - 1$ interchanges of adjacent indices in the list $\pi(1), \ldots, \pi(n)$ to get 1 to the first position in this list; once this has been done, it takes $\pi'(2) - 2$ more interchanges of adjacent indices to get 2 to the second position in the list, where $\pi'(2)$ the position of 2 in the new list, and so on. The upshot is that the coefficient of $p_\pi = a_(1\pi(1) \ldots a_{n\pi(n)}$ in either the row or column expansion of $\det A$ works out to be $(-1)^{\pi(1)-1+\pi'(2)-2+\ldots}$, and if π is multiplied on the right by $n(\pi) = \pi(1) - 1 + \pi'(2) - 2 + \ldots$ adjacent interchanges, we get the identity permutation, so that finally the coefficient of p_π in either a row or column expansion of $\det A$ is $\mathrm{sgn}(\pi)$, as required. In particular, the determinant of the 2×2 matrix $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is $a_{11}a_{22} - a_{12}a_{21}$, as we knew from before; similarly the determinant of the 3×3 matrix $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ is $a_{11}(a_{222}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$, again as we knew before. Also since $\det A$ can be computed by either a row or column expansion we see that any square matrix A has the same determinant as its transpose A^T.

We furthermore see that if a multiple of one row is added to another in A, then by multilinearity and the alternating property the determinant of the new matrix is the sum of two other determinants, one that of A and the other that of a matrix with two equal rows, so the new matrix has the same determinant as A; this row operation does not change the determinant. If two rows of A are interchanged, on the other hand, then $\det A$ does change, but only by a sign, so that in particular the determinant of the new matrix is 0 if and only if $\det A = 0$. Similarly, if a row of A is multiplied by a nonzero scalar, then $\det A$ is multiplied by the same scalar, so that again the determinant of the new matrix is 0 if and only if $\det A = 0$. Finally, once A is brought to reduced echelon form, its determinant is the product of its diagonal entries, this product being the only possibly nonzero term contributing to the above formula for $\det A$ in terms of permutations. But we know that A is invertible if and only if its reduced echelon form has the maximum possible number n of pivots, or equivalently the reduced echelon form is the identity matrix, so finally we see that we have achieved our main goal: $\det A = 0$ *if and only if* A *is singular.* As a side benefit, we learn along the way the most efficient method in practice for computing the determinant of A, namely to apply row operations as usual to bring it to echelon form (bearing in mind that interchanging two rows of A changes its determinant by a sign) and then take the product of the pivots in the echelon form, or take the determinant of A to be 0 if some row in the echelon form fails to have a pivot. We don't actually have to bring A all the way to *reduced* echelon form, since the determinant of any *triangular* matrix (having all zeroes either above the main diagonal, or all zeroes below this diagonal) is the product of the diagonal entries. Indeed, if all entries above the main diagonal of A are 0 (so that A is *lower triangular*) then expanding $\det A$ about the first row of A gives $(-1)^{1+1} = 1$ times a_{11} times the determinant $\det A^{11}$ of the 11-minor A^{11}, and A^{11} is again lower triangular whence by induction $\det A$ is the product of its diagonal entries, as claimed. The argument is similar for upper triangular matrices, expanding all determinants about the first columns.