Lecture 4-20

We now develop the general theory of determinants of $n \times n$ matrices; our goal here is to attach a single number $\det A$ to such a matrix A that will determine (hence its name) whether or not A is invertible. It is best to think of this number as a function $f(A_1, \ldots, A_n)$ of the rows $A_1, \ldots A_n$ (rather than the entries) of A, so that $f(A_1, \ldots, A_n) = 0$ if and only if the vectors A_i are dependent.

We know that the A_i are dependent if and only if they fail to span $\mathbb{R}^n$ and that the span of the A_i is unaffected if a row operation is performed on A. We also know that the A_i are dependent if two of them are equal. So we seek a a function $f(A_1, \ldots, A_n)$ that is unchanged if A_i is replaced by $A_i + kA_j$ for some $k \in \mathbb{R}$ and that is 0 if two of the A_i are equal. The best way to accomplish the first of these properties, given the second, is to make f *multilinear*, that is, linear in each A_i separately if the other A_i are fixed (just as the product function $p(x_1 \ldots, x_n) = x_1 \cdots x_n$ is linear in each x_i if the other ones are fixed). The second property (that f should vanish whenever two of the A_i are equal) is called the *alternating* property; thus we are looking for an alternating multilinear real-valued function of the A_i.

Remarkably enough there is just one such function up to scalar multiple, so that the determinant function f is uniquely specified by the requirements that it be multilinear and alternating in the rows together with the prescribed value $f(\vec{e}_1, \ldots, \vec{e}_n) = \det I = 1$. To prove this, let $A = (a_{ij})$ be any $n \times n$ matrix, so that its ith row $A_i = \sum_j a_{ij}\vec{e}_j$. Then any multilinear function f of the A_i has $f(A_1, \ldots, A_n) = \sum_\sigma a_{1\sigma(1)} \ldots a_{n\sigma(n)} f(\vec{e}_{\sigma(1),\ldots,\vec{e}_{\sigma(n)}}$, where the sum runs over all functions σ from the index set $\{1, \ldots, n\}$ to itself. This is a huge sum with n^n terms, but fortunately most of them vanish if f is in addition alternating, for then $f(\vec{e}_{\sigma(1)}, \ldots, \vec{e}_{\sigma(n)}) = 0$ if $\sigma(i) = \sigma(j)$ for any indices $i \neq j$. What survives in the formula for $f(A_1, \ldots, A_n)$ is $\sum_\pi a_{1\pi(1)} \ldots a_{n\pi(n)} f(\vec{e}_{\pi(1)}, \ldots, \vec{e}_{\pi(n)})$, where π runs through all *permutations* of $\{1 \ldots, n\}$, that is, all 1-1 and onto functions from this index set to itself, or rearrangements of this set. Note that there are exactly $n!$ such permutations π, since one such is specified by its value at the index 1 (for which there are n choices), then its value at 2 ($n-1$ choices), and so on.

Given an multilinear alternating function $f(A_1, \ldots, A_n)$, suppose that $A_i = A_j = \vec{v} + \vec{w}$ for $\vec{v}, \vec{w} \in \mathbb{R}^n$. Then $f(A_1, \ldots, \vec{v}+\vec{w}, \ldots, \vec{v}+\vec{w}, \ldots, A_n) = 0 = f(A_1 \ldots, \vec{v}, \ldots, \vec{v}, \ldots) +$ $f(A_1, \ldots, \vec{w}, \ldots, \vec{v}, \ldots) + f(A_1, \ldots, \vec{v}, \ldots, \vec{w} \ldots) + f(A_1, \ldots, \vec{w}, \ldots, \vec{w}, \ldots) =$ $f(A_1 \ldots, \vec{v}, \ldots, \vec{w}, \ldots) + f(A_1, \ldots, \vec{w}, \ldots, \vec{v}, \ldots)$. We deduce that *any multilinear alternating function changes sign if any two of its arguments are interchanged.* Now it is well known and quite intuitive that any permutation of n indices is the product of interchanges of just two indices: given a line of n books on the shelf that you want to rearrange in some manner, you can just interchange whatever book you want to be leftmost with the current leftmost book, then whatever book you want to be to its right with the book currently to its right, and so on. Hence *any nonzero multilinear alternating function $f(A_1, \ldots, A_n)$ taking the value 1 at $(\vec{e}_1, \ldots, \vec{e}_n)$ takes a uniquely determined value ± 1 at any permutation $(\vec{e}_{\pi(1)}, \ldots, e_{\pi(n)})$ of $\vec{e}_1, \ldots, \vec{e}_n$).* But now the problem is that it is conceivable that some permutation π might simultaneously be a product of evenly many interchanges and oddly many interchanges; if so, then any multilinear alternating function f would have to take the value 0 at $(\vec{e}_{\pi(1)}, \ldots, \vec{e}_{\pi(n)})$, and in fact the only such function would be the 0 function.

Clearly this is a fate too horrible to contemplate; fortunately, we will see next time that there is no such permutation π. That is, *every permutation π is either the product of evenly many interchanges, or oddly many interchanges, but not both*. We call a product π of evenly many interchanges *even* as a permutation and say that its *sign* $\mathrm{sgn}(\pi) = 1$; otherwise we call π *odd* as a permutation and write $\mathrm{sgn}(\pi) = -1$. We will show that signs are well defined next time. For now we just note that any interchange of two indices is the product of *oddly*, many interchanges of *adjacent* indices, since it takes $j-i$ interchanges of adjacent indices to bring index $j > i$ to the ith position and then $j-i-1$ such interchanges to bring index i to the jth position. Hence *any multilinear function $f(A_1, \ldots, A_n)$ which changes sign if two adjacent A_i are interchanged is in fact alternating*. This is a useful technical observation for our analysis next time. Note also that it is quite easy to see that the identity permutation of two indices is even while the non-identity permutation of two indices is odd, since here is just one non-identity permutation of two indices, and interchange, and the square of this interchange is the identity.