Lecture 4-17

We give some examples to illustrate the general theory developed last time. Starting with our favorite matrix $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, we have already seen that A reduces to the echelon form $E = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{pmatrix}$. Consequently its rank is 2, since there are two pivots in the echelon form. A basis for its row space is given by the nonzero rows $(1, 2, 3), (0, = 3, -6)$ of its echelon form. Its column space also has dimension 2; a basis is given by the first two columns of A (since those are the ones with pivots in E), or actually in this case by any two columns of A. The nullity of A is 1, since the system $AX = \vec{0}$ winds up having one free variable. A basis for this space is given by the single vector obtained by setting the free variable x_3 equal to 1 and then solving for the other variables; this vector is $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$. The left nullspace also has dimension 1; it has as basis the same vector $(1, -2, 1)$ (this is a coincidence, though since the matrix A is square we know in advance that the nullspace and left nullspace have the same dimension).

For an example with a nonsquare matrix we look at another old friend, namely the matrix $B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$. This time the echelon form E is $\begin{pmatrix} 1 & 2 \\ 0 & -2 \\ 0 & 0 \end{pmatrix}$. The columns and row spaces both have dimension 2, the column space having as basis the columns of B while the row space has basis $(1, 2), (0, -2)$ (or any two of the rows of B). This matrix has full rank. its nullspace is $\vec{0}$ while its left nullspace has as basis the same vector $(1, -2, 1)$ as in the last example.

The notion of rank extends in a natural way to linear transformations; given one such, say $f : V \to W$, we say that the rank of f is the dimension of $f(V)$, the range of f on V (which we have already observed is a subspace of W). This coincides with the rank of the matrix of f with respect to any choice of bases B, B' of V and W; thus the rank of this matrix does not change, though the matrix does, if B and B' are replaced by different bases. Since the range of f is spanned $f(v_i)$ as the v_i run over a basis of V, we see that the rank of any transformation from V to W is at most the minimum of the dimensions of V and W, just as the rank of a matrix is at most the minimum of the numbers of its rows and columns. Now we can more fully explain a remark made earlier. Given an $m \times n$ matrix A and an $n \times m$ matrix B with $m < n$, the products AB and BA are both defined and square, the first of these being $m \times m$ and the second $n \times n$. The second product BA is then always singular, since its rank is the dimension of $BA\mathbb{R}^n$, which is at most $m < n$, since $A\mathbb{R}^n \subset \mathbb{R}^m$; on the other hand, it is perfectly possible for the other product AB to be nonsingular, or even the identity. A simple example is $A = \begin{pmatrix} 1 & 2 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus the matrix equation $AB = I$ does not imply $BA = I$ if A and B are nonsquare matrices. Moreover, given linear transformations f, g from an *infinite-dimensional* vector space V to itself, it is possible for fg to be the identity while gf is not: take the vector space P of polynomials in one variable x, take g to be definite integration from 0 to x, and take f to be differentiation. (As a nice exercise, write down the *infinite* matrices of f and g with respect to the natural basis $1, x, x^2, \ldots$ of P!)

Similarly we define the *nullity* of a linear transformation $f : V \to W$ to be the dimension of its kernel; this coincides with the nullity of the matrix of f with respect to any choice of bases of V and W.

Given the $m \times n$ matrix A of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ (say with respect to the standard bases, its $n \times m$ transpose A^T is the matrix of another transformation $g : \mathbb{R}^m \to \mathbb{R}^n$. It is natural to wonder how the transformation g could be described directly in terms of f. We will show how to do this later, when we get dot products into the linear algebra picture.

Finally, to complete our running discussion (for this week; we will say more about this later) of symmetric matrices brought to diagonal form by row and column operations, we recall that in order to do this, we had to assume at various points along the way that entries in the ii position are nonzero, so that they could serve as pivots. But in fact any symmetric matrix can be brought to diagonal form without having to make this assumption. To see this, suppose that when we first come to the ith column we find that the iith entry is 0 but some lower entry in this column, say the jth, is nonzero. Then we just add the jth row to the ith row, simultaneously adding the jth column to the ith column. Now the iith entry is nonzero and we can proceed with the ith column as before. Moreover, if all the lower entries in the ith column are 0, then we are done with the ith column and can proceed to the next column. The upshot is that *given any symmetric matrix M there is a product P of elementary matrices such that $PMP^T = D$ is diagonal; then M is positive definite if and only if D has positive entries along its diagonal, and similarly negative definite if and only if D has negative entries along its diagonal.* Thus our algorithm for determining whether a symmetric matrix is positive definite is now entirely general.