Lecture 4-16

We saw last time that an $n \times n$ matrix has independent columns if and only if it has independent rows if and only if it has independent columns, so that its row space is all of $\mathbb{R}^n$ if and only if its column space is $\mathbb{R}^n$. We now want to massively generalize this result and in the process attach an important number to a matrix, not necessarily square. We have previously defined the column space of a matrix as the span of its columns; we now similarly define its row space to be the span of its rows. We now claim that *the row and column spaces of any matrix A, not necessarily square, always have the same dimension*; we call this dimension the *rank* of the matrix. To see this, we bring A to echelon form by row operations, as usual. Any row operation on a matrix preserves its row space, so the row space of the echelon form is the same as that of A. But now the row space of the echelon form is clearly spanned by its pivot, or nonzero rows; these are moreover independent since the coefficient of the topmost pivot row in any dependence relation must be 0 in order to make the coordinate in the pivot position 0, this being the only row having a nonzero in that position; then the coefficient of the next topmost pivot row must also be 0 in any dependence relation, and so on. Hence the dimension of the row space of A equals the number of pivots in its echelon form.

But now the dimension of the column space also equals the number of pivots in the echelon form. This is a bit tricker to prove, since row operations do not generally preserve the column space; but we have seen that the dimension of the column space is the dimension of the space of vectors B such that the system $AX = B$ has a solution. Since row operations are reversible this space has the same dimension as the space of vectors C such that $EX = C$ has a solution, where E is the echelon form of A. From our algorithm for solving linear systems we learn that $EX = C$ has a solution if and only the coordinate of C corresponding to any non-pivot(=zero) row of E is 0; so the dimension of the space of such vectors C equals the number of pivots in E, as claimed. Since the column space of A is also the range of the linear transformation with matrix A, we also call it the range of A itself and sometimes denote it Ran A. It has as a basis those columns of A that have pivots in its echelon from E. A matrix A is said to have *full rank* if its rank is as large as possible; that is, its rank equals the minimum of the numbers of its rows and columns.

Thus even though the row and column spaces of A generally live in different vector spaces, they always have the same dimension. Later we will strengthen this result: *multiplication by A, when restricted to its row space, gives an isomorphism onto its column space*. After we prove this last fact we will use it to work out the best possible solution in an appropriate sense to an inconsistent system $AX = B$.

Besides the row and column spaces of A another important space attached to A is its *kernel*, or *nullspace*, consisting of all vectors X with $AX = \vec{0}$. The homogeneous system $AX = \vec{0}$ is always consistent; the number of its free variables equals the number of columns of A minus the number of pivots in its echelon form, or the number of columns of A minus its rank. We get a basis for this space by setting each free variable in turn to 1 and the others to 0 and then solving uniquely for the remaining variables. Hence *the dimension of Ker A, also called the nullity of A, equals the number of columns of A minus its rank*. Note carefully that it is the number of columns of A and not rows of A that appears in this formula; it holds for arbitrary rectangular matrices A, but not in general if "columns" is replaced by "rows".

Yet another subspace attached to A is its *left nullspace*, which is the kernel of A^T, or equivalently the space of *row* vectors X with $XA = \vec{0}$. The dimension of this space equals the number of rows of A minus its rank. This space is not as important as the nullspace of A and accordingly is not given a special name in the Treil notes.

If we have to solve several linear systems with the same coefficient matrix A, say $AX = B_1, \ldots, AX = B_m$ (not necessarily with the same solutions X), we can do it most efficiently by fattening up the usual augmented matrix: add columns $B_1, \ldots, V_m$ to the right of the matrix A to make a larger matrix. Then we can bring A to echelon form, taking the extra columns B_i along for the ride, and so simultaneously solve the systems. In particular, if A is $n \times n$ and invertible, then in order to find its inverse A^{-1}, we must solve the systems $AX = \vec{e}_1, \ldots, AX = \vec{e}_n$ to produce the columns of A^{-1}. Hence *we can compute A^{-1} by adding a copy of I, the identity matrix, to the right of A, and then row-reducing A to I; the same row reductions applied to I, then give A^{-1}.*

Finally, continuing with the procedure given earlier for bringing a symmetric matrix M to diagonal form by row and column operations, we observe that the row operations are implemented by multiplying M on the left by a suitable product $E_n \cdots E_1$ of elementary matrices; at the same time we multiply M on the right by the reverse product $E_1^T \cdots E_n^T$ of their transposes, so as to implement the column operations. Thus if D is the diagonal matrix obtained by these row and column operations, then we have $D = E_n \cdots E_1 M E_1^T \cdots E_n^t$. But now for every row vector $\vec{v}$ we have $\vec{v} D \vec{v}^T = (\vec{v} E_1 \cdots E_n) M (E_n^T \cdots E_1^T \vec{v}^T) = \vec{w} M \vec{w}^T$, where $\vec{w} = \vec{v} E_1 \cdots E_n$, since the transpose of a product is the reverse product of the transposes. Hence it is indeed the case, as mentioned earlier, that D is positive definite (or negative definite) if and only if M is, for symmetric matrices M.