Lecture 4-15

Continuing from last time, we now want to see how to bring a matrix in echelon form to reduced echelon form. Start with our first old friend, the matrix $M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{pmatrix}$. We have already seen that an echelon form of this matrix is $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 1 \end{pmatrix}$. Now we make things a little easier for ourselves by first dividing the second row by -3, making the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Next we subtract twice the second row from the first one, obtaining $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Next, we add the third row to the first one and at the same time subtract twice the third row from the second one, obtaining finally $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which is the identity matrix I. Had we started with the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ in our second example, we would have wound up with the matrix $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$; had we started with our first rectangular matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, we would have wound up with $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}$. In all three cases you can check that the matrix we got in the end really is in reduced echelon form.

Now the only possible *reduced* echelon form of a square matrix in which every row has a pivot is the identity matrix I. If a square matrix M can be brought to reduced echelon form using our three row operations (adding a multiple of one row to another, interchanging two rows, or multiplying a row by a nonzero scalar), then there are elementary matrices $E_1, \ldots, E_m$ such that $E_1 \cdots E_m M = I$, whence one easily checks that $M = E_m^{-1} \cdots E_1^{-1}$. Here it is easy to see that any elementary matrix is invertible; the inverse matrix is the matrix of the inverse operation (adding a times row j to row i if the original operation subtracted a times row j from row i, interchanging rows i and j if the original operation did the same, and finally multiplying row i by $a \neq 0$ if the original operation divided row i by a). Hence given any square matrix M with linearly independent columns, all of its rows have pivots when it is brought to echelon form, whence the only possible reduced echelon form of it is I, and M is invertible (being the product of invertible elementary matrices). Conversely, if M is square but does not have independent columns, then there is a nontrivial solution to $MX = \vec{0}$, whence the same is true if M is replaced by a matrix in reduced echelon form obtained from it by row operations, whence no sequence of row operations can transform M to I. Moreover M is not invertible in this case, since if it were the only solution X to the system $MX = \vec{0}$ would be $X = M^{-1}MX = M^{-1}\vec{0} = \vec{0}$. We conclude that *a square matrix M is invertible if and only if its columns are linearly independent*. Likewise, *such a matrix M is invertible if and only if its rows are linearly independent*. Indeed, the rows of a matrix are linearly dependent if and only if the same is true of the rows of any matrix obtained from by a row operation, since any such operation can be inverted and replaces the rows of M by linear combinations of themselves. Hence the only possible reduced echelon form of a square matrix with independent rows is I and such a matrix is invertible, while conversely we have seen that an invertible matrix has reduced row echelon form I and so must have independent rows. Also a square matrix is invertible if and only if it is a product of elementary matrices.

We conclude with an easy but useful operation on matrices and an important number attached to a square matrix. Given an $m \times n$ matrix $A = (a_{ij})$, its *transpose* A^T is the $n \times m$ matrix whose ji-th entry is a_{ij}; thus we write down A^T by writing the rows of A as the columns of a new matrix, Given a product AB of matrices that is defined, we know that the ij-th entry of AB is the dot product of the ith row of A and jth column of B; since these coincide with the ith column of A^T and jth row of B^T, respectively, we have $(AB)^T = B^T A^T$, so the transpose of a product is the reverse product of the transposes. The *trace* tr A of an $n \times n$ matrix A is the sum $\sum_{i=1}^n a_{ii}$ of its diagonal entries; trace is undefined for nonsquare matrices. Given an $n \times m$ matrix A and an $m \times n$ matrix B, so that both products AB, BA are defined and square (but generally of different sizes), we have $\text{tr } AB = \sum_{i=1}^n \sum_{j=1}^m a_{ij} b_{ji} = \text{tr } BA$. We will work out consequences of this formula later. For now, let's work out an example of the procedure outlines last time to bring a symmetric matrix to diagonal form by row *and* column operations. Start with the symmetric matrix $M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Subtract row 1 from row 3 and simultaneously column 1 from column 3, to produce the matrix $M' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$; note that (as promised last time) M' is still symmetric. Now subtract row 2 from row 3 and simultaneously column 2 from column 3, to produce the diagonal matrix $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Clearly D is *not* positive definite, having a -1 on the main diagonal; so M is not positive definite either.