Lecture 4-13

Last time we saw that any linear map $f : V \to W$ between a pair V, W of finite-dimensional vector spaces has a unique matrix M relative to any choice of bases $B.B'$ of V, W, respectively. In particular, if $V = \mathbb{R}^n, W = \mathbb{R}^m$, then by far the most common choices for B and B' are the standard bases, whose i-th vectors in both cases are the i-th unit coordinate vector, having 1 as the i-th coordinate and 0 as the other coordinates. Fixing these bases, we see that *any linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is given by left multiplication of a column vector in $\mathbb{R}^n$ by an $m \times n$ matrix (producing thereby a column vector in $\mathbb{R}^m$).* The correspondence between linear transformations and matrices becomes even more useful when one observes that *given two transformations $f : \mathbb{R}^n \to \mathbb{R}^m, g : \mathbb{R}^m \to \mathbb{R}^p$, with respective matrices A, B, so that A is $m \times n$ and B is $p \times m$, the composite transformation $gf : \mathbb{R}^n \to \mathbb{R}^p$, which is easily seen to be linear, has BA (a $p \times n$ matrix) as its matrix*; this follows at once from the definitions of matrix multiplication and linear map. As an immediate (and highly non-obvious) consequence we see that *matrix multiplication is associative whenever defined*, as mentioned earlier. That is, to be more precise, a triple matrix product $(AB)C$ is defined if and only if $A(BC)$ is defined and then the two products are equal. This follows since if A, B, C respectively represent the maps f, g, h, then either of these triple products represents the composite $(f(gh) = (fg)h$. On the other hand, even if we focus attention on linear maps from $\mathbb{R}^n$ to itself, two such maps generally fail to commute if $n > 1$, so once again we see that multiplication even of $n \times n$ matrices for fixed $n > 1$ is not commutative.

A weak kind of commutativity that does hold, however, is the following: given a linear map f from $\mathbb{R}^n$ to itself that is 1-1 and onto, so that its inverse f^{-1} is well-defined as another map from $\mathbb{R}^n$ to itself, this inverse is easily seen to be linear. The composite $ff^{-1} = f^{-1}f$ of the two maps in either order is then the identity map, whose matrix is the identity matrix I (having 1s on the *main diagonal* from upper left to lower right and 0s elsewhere). Hence *for two $n \times n$ matrices A, B we have $AB = I$ if and only if $BA = I$*; we write $B = A^{-1}$ in this case and call A *invertible* or *nonsingular*. (Note however that if A and B are $n \times m$ and $m \times n$ respectively for some $n \neq m$ then it is quite possible that $AB = I$ but $BA \neq I$; in fact, we will see later that it is not possible BA to equal I in this situation if $m > n$.) Moreover, if f takes $\mathbb{R}^n$ to $\mathbb{R}^n$, it is not necessary to check that f is *both* 1-1 and onto: we have seen that a square matrix A has pivots in all rows when brought to echelon form if and only if it has pivots in all columns in that form and the columns of A are dependent if and only if they span $\mathbb{R}^n$, so that a linear map with this matrix is 1-1 if and only if it is onto. We will see later that there is a single number attached to a square matrix A called its *determinant* that is nonzero if and only if A is nonsingular.

A word of caution here: the correspondence between linear maps between a pair of fixed finite-dimensional vector spaces and matrices of a fixed size depends on the choice of fixed bases for the vector spaces; it changes if those bases are changed. We will explore exactly how the matrix changes if the bases are changed later. For now let me mention that, by a standard convention, if we look at linear transformations f from a finite-dimensional vector space V to itself, then we usually choose just one basis B of V and insist on using this basis for both the domain and range of f in computing its matrix. If the square matrices M, N represent the same transformation $f : V \to V$ with respect to different bases, then we call M and N *similar*; we will see later that this is a very interesting relationship between matrices arising in many contexts. We can see already that a lot has to be similar (in the ordinary English sense) about similar matrices; for example, any matrix similar to an invertible one has to be again invertible (representing as it does an invertible transformation). Now recall that a square matrix M is said to have *eigenvalue* λ and *corresponding eigenvector* $V \neq 0$ if $MV = \lambda V$. What can be said about a matrix N similar to M in this situation? You can't quite say that $NV = \lambda V$, since the n-tuple representing V comes from the coefficients of the basis vectors in a particular combination equalling V and those coefficients could change if the basis is replaced by a different one. But you *can* say that $NW = \lambda W$ for *some* $W \neq 0$: *similar matrices have the same eigenvalues*. We will say a lot more about eigenvalues and eigenvectors (and similar matrices) later.