Lecture 4-10

Last time we showed that any generating set of a finite-dimensional vector space V can be shrunk to a basis of V. Similarly, if we start with an independent subset $v_1, \ldots, v_m$ of V we can enlarge it to a basis of V, simply by adding a basis $w_1, \ldots, w_n$ of V to the original list $v_1, \ldots, v_m$ and running through the resulting list $v_1, \ldots, v_m, w_1, \ldots, w_n$, eliminating vectors that are linear combinations of preceding vectors, until we obtain a basis of V; we never eliminate any v_i during this procedure, since no v_i is a combination of preceding ones. We also see that *every subspace W of V is also finite-dimensional and in fact of dimension at most that of V and the dimension of W equals that of V if and only if $W = V$*. This follows since no independent subset of W can have more than $\dim V$ (the dimension of V) vectors, whence there is a maximal subset M of independent vectors in W (one not properly contained in any other). This subset must then also be a basis of W, since if any $w \in W$ is not in its span, we could add w to M to produce a larger independent subset, contradicting the way M was chosen. If $\dim W = \dim V$, then any basis of W is still independent in V, whence it spans V by previous results, forcing $V = W$. (Note however that *infinite-dimensional* vector space V can have proper subspaces W of the same dimension, e.g. the subspace of all linear combinations of odd powers $1, x, x^3, \ldots$ of the variable x in the space of polynomials in x. This same example shows that a countable independent subset of a countable dimensional space need not span the space; similarly a countable spanning subset need to be independent in such a space.)

Now we are finally ready to make precise our earlier informal statement that a subspace of $\mathbb{R}^n$ looks just like $\mathbb{R}^m$ for some $m \leq n$. Any such subspace S has dimension at most n and thus a basis $\vec{v}_1, \ldots, \vec{v}_m$ for some m. Now we have a bijection f from $\mathbb{R}^m$ onto S sending the m-tuple $(a_1, \ldots, a_m)$ to the combination $\sum a_i v_i$. The definition of basis guarantees that this map is indeed a bijection. It is a lot more than just a set-theoretic bijection, however, since we have $f(\vec{v} + \vec{w}) = f(\vec{v}) + f(\vec{w}), f(k\vec{v}) = kf(\vec{v})$ for all $\vec{v}, \vec{w}, \in \mathbb{R}^m, k \in \mathbb{R}$. We have seen such maps before and called them *linear*; we now officially decree that any map $f : V \to W$ between a pair V, W of vector spaces is called linear whenever it satisfies these properties. A linear map between vector spaces that is also a bijection is called an *isomorphism* and the vector spaces are called *isomorphic*; they can then be regarded as essentially the same for most purposes. Thus *any* finite-dimensional (real) vector space V is isomorphic to $\mathbb{R}^n$ for $n = \dim V$ (and not to $\mathbb{R}^m$ for any $m \neq \dim V$).

Probably the simplest example of an infinite-dimensional vector space is the set of polynomials in one variable x; here a basis is given by $1, x, x^2, \ldots$ You might think that this same set would be a basis for the much larger set of functions analytic at 0, since any such function $f(x)$ by definition admits a power series expansion $\sum a_n x^n$ that converges to $f(x)$ at least for $|x| < R$ for some $R > 0$. Recall however that vectors in a vector space must be *finite* linear combinations of basis vectors; we never consider infinite series of vectors in linear algebra. Thus the space spanned by $1, x, x^2, \ldots$ indeed consists of polynomials in x and nothing else.

Now let V, W be any two vector spaces over $\mathbb{R}$ (not necessarily finite-dimensional). A function (or map) f from V to W is called *linear* if $f(v_1 + v_2) = f(v_1) + f(v_2), f(rv) = rf(v)$ for all $v, v_1, v_2 \in V, r \in \mathbb{R}$. If V happens to be finite-dimensional, say with basis $v_1, \ldots, v_n$, and if $w_1, \ldots, w_n$ are any n vectors in W, then there is a unique linear map sending v_i to

w_i for all i; more generally, it sends any linear combination $\sum r_i v_i$ to the corresponding combination $\sum r_i w_i$ of the w_i. (One easily checks that the map defined by this last formula is indeed linear.) Thus a linear map is completely determined by what it does to a basis, and what it does to that basis is arbitrary. Now suppose in addition that W is finite-dimensional, say with basis $w_1, \ldots, w_m$. Given a linear map f from V to W, we have $f(v_i) = \sum j = 1^m a_{ji} w_j$ for some $a_{ji} \in \mathbb{R}$, where the indices i, j run from 1 to n and 1 to m, respectively. You can probably guess from the notation I have used what my next step is: define a matrix A whose ij-th entry is a_{ij}. We call this the *matrix of f with respect to the bases $\{v_1, \ldots, v_n\}, \{w_1, \ldots, w_m\}$ of V, W*. Given now a vector $v = \sum i = 1^n r_i v_i$ in V, make a column vector $\vec{v}' = \begin{pmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_n \end{pmatrix}$ and write $A\vec{v}' = \vec{w}' = \begin{pmatrix} s_1 \\ \cdot \\ \cdot \\ \cdot \\ s_m \end{pmatrix}$. Then you can check that the combination $\sum_{j=1}^m s_j w_j$ is exactly the image $f(v)$ of v under f. Hence, once bases B, B' of two finite-dimensional vector spaces V, W are fixed, and we set up an $m \times n$ matrix whose ith column consists of the coefficients of the vectors in B' in the image $f(b_i)$ of the ith vector b_i in B, the resulting matrix is the matrix of f with respect to B and B'. An arbitrary combination of vectors in B' lies in the range of f if and only if its coefficients form a column vector spanned by the columns of this matrix; we call this span the *column space* of the matrix.