Lecture 3-9

We begin our review with the most recent material on differentiability. We give a slightly different (but equivalent) definition: a function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\vec{a} \in \mathbb{R}^n$ if there is a constant vector $\vec{d} = (d_1, \ldots, d_n)$ with $\lim_{\vec{h} \to \vec{0}} \frac{(f(\vec{a}+\vec{h}) - f(\vec{a}) - \vec{d} \cdot \vec{h}}{||\vec{h}||} = 0$. If such a vector $\vec{d}$ exists, then it must be the gradient vector $\nabla f(\vec{a})$; in particular, all partials of f must exist at $\vec{a}$. If these partials exist in a neighborhood of $\vec{a}$ and in addition all are continuous at $\vec{a}$, then f is always differentiable there; we say that f is continuously differentiable at $\vec{a}$ in this situation. As a tricky exercise, show that the function $f(x, y) = x^2 + y^2$ if $x, y \in \mathbb{Q}, f(x, y) = 0$ otherwise, is actually differentiable at $(0, 0)$. If f is differentiable at $\vec{a}$, then a great many rates of change of f at $\vec{a}$ can be read off from $\nabla f(\vec{a})$; in particular, for any unit vector $\vec{u}$, the directional derivative of f in the $\vec{u}$ direction, defined to be $\lim_{t \to 0} \frac{f(\vec{a}+t\vec{u}) - f(\vec{a})}{t}$, is given by $\nabla f(\vec{a}) \cdot \vec{u}$; more generally, given any parametrized curve $\vec{r}(t)$ passing through $\vec{a}$ at time $t = t_0$, the composite function $f(\vec{r}(t))$ is also differentiable at t_0 and its derivative there is $\nabla f(\vec{a}) \cdot \vec{r}'(t_0)$. As a consequence, any parametrized curve $\vec{r}(t)$ lying in a level set $S = \{\vec{x} : g(\vec{x}) = c\}$ for some differentiable function g and constant c has $\nabla g(\vec{a}) \cdot \vec{r}'(t_0) = 0$ if $\vec{r}(t_0) = \vec{a}$. For this reason, whenever $\nabla g(\vec{a}) \neq \vec{0}$ for some $\vec{a} \in S$, we call the hyperplane passing through $\vec{a}$ with normal vector $\nabla g(\vec{a})$ the tangent hyperplane to the level set S at $\vec{a}$; if instead $\nabla g(\vec{a}) = \vec{0}$, then we say that the tangent hyperplane to S at $\vec{a}$ is undefined.

Returning now to the first week of the course, we recall the standard parametrizations of the conic sections; the ellipse in standard position with equation $\frac{x^2}{z^2} + \frac{y^2}{b^2} = 1$, it is parametrized by $x = a \cos t, y = b \sin t$, while the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is parametrized by $x = \pm a \cosh t, y = b \sinh t$ (we could also set $x = a \sec t, y = b \tan t$). The parabola $y^2 = 4cx$ is of course parametrized via $x = t^2/(4c), y = t$. A point $(x(t), y(t)$ of a general parametrized curve in the plane has tangent line of slope $y'(t)/x'(t)$ whenever $x'(t) \neq 0$; the arclength of the curve segment corresponding to the interval $[a, b]$ for t is $\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$. The curve is said to be parametrized by arclength if the length $\sqrt{x'(t)^2 + y'(t)^2}$ of the tangent vector is constantly equal to 1; we use the letter s to denot the arclength parameter. The curvature of a plane curve at $(x(t), y(t))$ (the amount it bends per unit of arclength) is given by the formula $\frac{|x'(t)y''(t) - y'(t)x''(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}}$. A closed curve segment $(x(t), y(t))$ for $t \in [a, b]$ (such that $(x(a), y(a)) = (x(b), y(b))$) encloses a region of area $\int x\, dy = \int_a^b x(t)y'(t)\, dt$, provided that $(x(t), y(t))$ traces the boundary of this region counterclockwise as t increases from a to b.

More generally, a parametrized curve $\vec{r}(t) = (r_1(t), \ldots, r_n(t))$ in $\mathbb{R}^n$ has tangent vector $\vec{r}'(t) = (r_1'(t), \ldots, r_n'(t))$ at any point; the coordinate functions $r_i(t)$ must be differentiable by definition of a parametrized curve. (Note in this case that the notion of the slope of a line makes no sense in $\mathbb{R}^n$ for $n > 2$, so the earlier formula $y'(t)/x'(t)$ does not carry over.) The arclength of the curve segment corresponding to the interval $[a, b]$ for t is given by $\int_a^b ||\vec{r}'(t)||\, dt$, as for plane curves, and we say as before that $\vec{r}(t)$ is parametrized by arclength if $||\vec{r}'(t)|| = 1$ for all t. In any event, the *unit tangent vector* $\vec{T}(t)$ is defined to be $\vec{r}'(t)/||\vec{r}'(t)||$, provided that $\vec{r}'(t) \neq \vec{0}$; since we then must have $\vec{T}(t) \cdot \vec{T}'(t) = 0$, we call the unit vector $\vec{T}'(t)/||\vec{T}'(t)||$ the *principal unit normal* (again provided that $\vec{T}'(t) \neq \vec{0}$). The curvature of $\vec{r}(t)$ is given by $||d\vec{T}/ds||$; the derivative of the unit tangent vetor $\vec{T}$ with respect to arclength s; by the chain rule this may also be computed as $||(d\vec{T}/dt)/(ds/dt)||$, where $ds/dt = ||\vec{r}'(t)||$, as mentioned above.

For a general curve $\vec{r}(t)$, not necessarily parametrized by arclength, we have the respective formulas v' and κv^2 for the tangential and normal components of acceleration; that is, for the coefficients of the unit tangent $\vec{T}(t)$ and principal unit normal $\vec{N}(t)$ in the formula for the acceleration vector $\vec{a}(t) = \vec{r}''(t)$; here the scalar v denotes the speed $||\vec{r}'(t)||$. If $n = 3$, so that $\vec{T}(t)$ and $\vec{N}(t)$ live in $\mathbb{R}^3$, the cross product $\vec{B}(t)$ is called the (unit) binormal vector.

An important example of a parametrized curve is the cycloid of radius a; this is the path traced by a point on a circumference of a wheel of radius a that rolls without slipping down the x-axis in the positive direction, starting at $(0, 0)$. The equations are $x(t), y(t)) = (a(t - \sin t), a(1 - \cos t)$ and the length of one complete arch of this cycloid is $8a$.

Turning now to sequences and series, we recall that a power series $\sum a_n x^n$ has a radius of convergence R which captures the values of x for which the series converges; more precisely, the series converges if $|x| < R$ and diverges if $|x| > R$ (so that it always converges if $R = \infty$ but converges only for $x = 0$ if $R = 0$). The radius of convergence R is usually given by the ratio test: we have $R = \lim_{n\to\infty} |a_n/a_{n+1}|$ whenever the limit exists. More generally, the same is true of power series $\sum a_n(x - a)^n$ in $x - a$: these too have radii of convergence R, so that they converge if $|x - a| < R$ and diverge if $|x - a| > R$.