Lecture 3-5

We now move on to some topics from Chapter 15; you may want to skim this chapter at some point, though the material in it will not appear on the final exam.

We studied level sets of functions of n variables last time, showing that such sets have well-defined tangent hyperplanes at most points (where the gradient of the defining function is not 0). We mention here that level curves (of functions of two variables) and surfaces (of functions of three variables) occur quite often in real life; for example, a topographical map typically has curves of constant altitude drawn on it, so that the altitude of any point corresponding to a point on the map is given by the number attached to the altitude curve passing through that point, with different points on the same altitude curve having the same altitude; similarly one finds isobars (curves of equal pressure) and isotherms (curves of equal temperature) on many weather maps, and so on.

We digress from functions to say a few words about subsets of $\mathbb{R}^n$. A subset S of $\mathbb{R}^n$ is called *open* if for every $\vec{a} \in S$ there is $\epsilon > 0$ such that every point at distance less than ϵ from $\vec{a}$ also lies in S; the empty subset of $\mathbb{R}^n$ is also (trivially) called open. Contrary to what you might expect from the terminology, a closed subset is *not* simply one that is not open; in fact most subsets of $\mathbb{R}^n$ are neither open nor closed! We define a *boundary point* of a set S to consist of all $\vec{x} \in \mathbb{R}^n$ such that for every $\epsilon > 0$ there are points within ϵ of $\vec{x}$ lying in S and other such points not lying in S. The boundary of S is the set of all of its boundary points. Note that a boundary point of a set may or may not lie in the set; for example, the boundary of the Cartesian product $[0, 1) \times [0, 1]$ of intervals consists of the two line segments from $(0, 0)$ to $(0, 1)$ and from $(1, 0)$ to $(1, 1)$ in the plane; the first of these segments lies in S but the second is disjoint from it. Then a set S is called *closed* if its contains all of its boundary points. (Conversely, a set is open if and only if it contains none of its boundary points.) Roughly speaking, if we picture a subset S of say $\mathbb{R}^2$ via a blob drawn on a sheet of paper, then the set is closed if and only if the blob is drawn with solid boundaries, indicating that the points on these boundaries lie in S. Another good rule of thumb is that a subset is defined by one ore more strict inequalities is open, while one described by one or more weak inequalities is closed. Thus the "open unit disk" $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ is indeed open, while the ordinary unit disk $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ is closed. Any level set of a continuous function is closed, since it consists of the points where that function is equal to a fixed value. The only subsets of $\mathbb{R}^n$ that are both open and closed are the empty set $\emptyset$ and $\mathbb{R}^n$ itself.

It is worth making these definitions of open and closed subsets because when we come to deciding where real-valued functions f defined on subsets S of $\mathbb{R}^n$ have local maxima or minima, the nature of S makes a big difference. If S is open, so that given any $\vec{a} \in S$ we are free to approach $\vec{a}$ from any direction and stay in S (so long as we are sufficiently close to $\vec{a}$), then you will not be surprised to learn that a necessary condition for f to have a local maximum or minimum at $\vec{a}$ is that all partial derivatives $\partial f/\partial x_i$ are either 0 or undefined at $\vec{a}$. This follows because a short enough line segment parallel to any coordinate axis with center $\vec{a}$ will lie entirely in S; if we parametrize this segment in an obvious way, so that $\vec{a}$ corresponds to the value $1/2$ of the parameter t, then when f is restricted to this parametrized segment, it must have a local maximum or minimum at $t = 1/2$, whence the partial derivative of f with respect to the variable corresponding to the coordinate axis must be either undefined or 0 at $\vec{a}$. If instead S is closed, however, then this argument breaks down and we need different techniques to decide when a point is a candidate for being a local maximum or minimum. For example, given the function $f(x, y) = xy$ restricted to the unit circle in the xy-plane, we note that a typical point on this circle takes the form $(\cos t, \sin t)$ for some $t \in \mathbb{R}$ and $f(\cos t, \sin t) = \cos t \sin t = (1/2) \sin 2t$. Hence f is maximized at the point $(\sqrt{2}/2, \sqrt{2}/2)$, corresponding to $t = \pi/4$; but clearly neither partial derivative of f vanishes at that point. It turns out that there is no general method for finding local maxima and minima of functions defined on arbitrary closed subsets, but if these subsets happen to be level sets of functions (as they often are in practice) then there is such a method, called that of *Lagrange multipliers*. We will learn about Lagrange multipliers next term.

For now we just observe that the boundary of a subset S of $\mathbb{R}^n$ is by definition as the boundary of its complement S^c (consisting by definition of all points of $\mathbb{R}^n$ not in S). It follows that a subset S is closed (contains all of its boundary points) if and only if its complement is open (contains none of them). This then is the actual relationship between open and closed subsets: a closed subset is not one that is not open, but rather one whose complement is open. Thus our earlier observation (not proved) that the only subsets of $\mathbb{R}^n$ that are both open and closed are $\emptyset$ and $\mathbb{R}^n$ itself is equivalent to the observation that the only open subsets of $\mathbb{R}^n$ whose complements are also open are $\emptyset$ and $\mathbb{R}^n$. We express this property by saying that $\mathbb{R}^n$ is *connected*. Many subsets S of $\mathbb{R}^n$ are both connected and *path-connected*, in the sense that for any $\vec{a}, \vec{b} \in S$ there is a parametrized curve segment starting at $\vec{a}$, ending at $\vec{b}$, and lying entirely in S. Any continuous real-valued function f on a path-connected subset satisfies the Intermediate Value Property, for if f takes the respective values c, d at $\vec{a}, \vec{b} \in S$, then by composing f with a suitable parametrized curve we get a continuous function on an interval on the real line, which we already know satisfies the IVP.