Lecture 3-13

We wind up our review with an account of level sets and tangent (hyper)planes. Given a differentiable function $g : \mathbb{R}^n \to \mathbb{R}$ the set $S = \{(x_1, \ldots, x_n \in \mathbb{R}^n : g(x_1, \ldots, x_n) = c\}$ is called a level set of g (here c is a constant). As we will have many occasions to do calculus on functions defined only on level sets rather than open sets in $\mathbb{R}^n$, we would like such sets to look as much like $\mathbb{R}^{n-1}$ as possible; to that end, we observe that if $\vec{r}(t)$ is a parametrized curve lying entirely in S and passing through $\vec{a} \in S$, say at $t = t_0$, then $\nabla f(\vec{a}) \cdot \vec{r}'(t_0) = 0$, by the chain rule, whence the unique hyperplane passing through $\vec{a}$ with normal vector $\nabla f(\vec{a})$ (assuming that this vector is nonzero) contains the tangent vector $\vec{r}'(t_0)$ at $t = t_0$ of any such curve. Accordingly we call this (hyper)plane the tangent (hyper)plane to S at $\vec{a}$ (if instead $\nabla f(\vec{a}) = \vec{0}$, then we call $\vec{a}$ a singular point of S and decree that the tangent hyperplane of S at that point is undefined).

A particular example where the tangent hyperplane is always defined is the case where the level set is the graph of a differentiable real-valued function $x_n = f(x_1, \ldots, x_{n-1})$. This graph can be viewed as the level set of the function $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_{n-1}) - x_n$ corresponding to $c = 0$; since $\partial g/\partial x_n = -1 \neq 0$ at any point of this level set, the gradient ∇g is never $\vec{0}$ on it, so that there is indeed a tangent hyperplane at every point. The equation of this hyperplane at $(a_1, \ldots, a_{n-1}, a_n)$, where $a_n = f(a_1, \ldots, a_{n-1})$, is $x_n - a_n = \sum_{i=1}^{n-1} (\partial f/\partial x_i)(a_1, \ldots, a_{n-1}(x_i - a_i$. In particular, if we use this hyperplane to estimate the value of f at a point $\vec{a}+\vec{h}$ near $\vec{a} = (a_1, \ldots, a_{n-1})$, then we get $f(\vec{a})+\nabla f(\vec{a})\cdot\vec{h}$, a fundamental term in the numerator of the limit definition of differentiability. Writing it out more explicitly, we have that $f(a_1 + h_1, \ldots, a_{n-1} + h_{n-1})$ is approximately equal to $f(a_1, \ldots, a_{n-1} + (\partial f/\partial x_1)(a_1, \ldots, a_{n-1})h_1 + \ldots + (\partial f/\partial x_{n-1})(a_1, \ldots, a_{n-1})h_{n-1}$. This is the multidimensional version of linear approximation.

Good luck on Wednesday. My bear is right here on my desk sending all good vibrations your way, as is Ethan's pi pillow (from his desk).