Lecture 3-11

Continuing with differential equations, suppose now that we have a nonhomogeneous equation $y'' + p(t)y' + q(t)y = r(t)$ for some nonzero function $r(t)$. The first principle is that the general solution to such a equation is the sum of one particular solution $y_p(t)$ and the general solution to the corresponding homogeneous equation $y'' + p(t)y' + q(t)y = 0$, so we are reduced (at least in theory) to finding just one solution $y_p(t)$. To do this we can either use inspired intelligent guessing (called the method of undetermined coefficients), which sometimes requires that we look further afield for solutions than we might have though at first (thus for example we might need to multiply a proposed solution by some power of t or by $\ln t$ to make it work), or else the method of variation of parameters, which provides a uniform recipe for y_p if two independent solutions y_1, y_2 to the homogeneous equation are known. In this case, one can set $v_1 = -\int (ry_2/W)\,dt, v_2 = \int (ry_1/W)\,dt$, where $W = y_1(t)y_2'(t) - y_2(t)y_1'(t)$ is the Wronskian of the two solutions y_1, y_2 to the homogeneous equation; then the particular solution is $y = v_1y_1 + v_2y_2$. As you all learned from the midterm, in order for these last formulas for v_1 and v_2 to work, it is essential that the coefficient of y'' in the given equation $y'' + p(t)y' + q(t)y = r(t)$ be 1 (otherwise one must divide by this coefficient).

Laplace transforms are the last method we saw in the course for solving initial-value problems (without having to generally solve the equations coming from these problems), thanks to the general formula $\mathcal{L}f^{(n)}(s) = s^n\mathcal{L}f(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$ for the transform of the nth derivative of a function f in terms of the transform of f itself and its initial values at time $t = 0$. Of course other particular formulas, such as $p!/s^{p+1}, 1/(s-a), a/(s^2+a^2), s/(s^2+a^2)$, for the respective transforms of $t^p, e^{at}, \sin at$, and $\cos at$ are important as well; we also recall that the transform of $u_c(t)f(t-c)$, the function whose graph is obtained from that of f by shifting it c units to the right and then adding a line segment from $(0,0$ to $(c,0)$, is $e^{-cs}F(s)$, where F is the transform of f. (More generally, the transforms of $e^{at}\sin bt$ and $e^{at}\cos bt$ are $\frac{b}{(s-a)^2+b^2}$ and $\frac{s-a}{s-a)^2+b^2}$, respectively.) The basic idea in solving any equation by Laplace transforms is simple enough: take the transform of both sides, solve for the transform of the unknown function, and then take the inverse transform to solve for the function itself. Although there is no formula for the transform of the product of two functions in terms of the transforms of the functions, there is a formula for the *inverse* transform of a product. It states that the transform of the convolution $\int_0^t f(t-\tau)g(\tau)\,d\tau$ is the product $F(s)G(s)$ of the transforms F, G of f, g. We can use this formula in two ways, either to recognize a certain integral as a convolution and so be able to take its Laplace transform, or to express the solution to a differential equation in terms of a convolution integral. Also occasionally useful is the formula $F^{(n)}(s)$, the nth derivative of the transform F of f, for the transform of $(-t)^n f(t)$.

The other main topic from differential equations covered this quarter is the existence-uniqueness theorem, which states that the initial-value problem $y' = f(t,y), y(t_0) = y_0$ always has a unique solution whose graph exists up to the boundary of a rectangle $R = [a,b] \times [c,d]$, provided that $(t_0, y_0) \in (a,b) \times (c,d)$ is an interior point of R and f and $\partial f/\partial y$ exist and are continuous on R. (If we assume only that f is continuous on R, then we can still say that a solution exists to this problem, though it might not be unique.) If the equation is not given in the form $y' = f(t,y)$, typically because the coefficient of y' in it is something other than 1, then one must divide by this coefficient and thus worry about points where this coefficient is 0. At such points existence or uniqueness of the solution might fail.

A differential equation of the form $M\,dt + N\,dy = 0$ (equivalent to $dy/dt = -M/N$) is called exact if there is a function $f(t, y)$ such that $\partial f/\partial t = M, \partial f/\partial y = N$. In this case we now know that the general solution to the equation is $f(t, y) = c$ for some constant c (by the chain rule). In particular any separable differential equation (one for which M depends on t alone while N depends only on y) is exact; the general criterion for $M\,dt + N\,dy = 0$ to be exact is that $\partial M/\partial y = \partial N/\partial t$ (if M, N are assumed to be defined on a rectangle, for example.)