Lecture 3-10

Continuing from last time, we recall two fundamental results on power and Taylor series, namely that they can be differentiated or integrated term by term within the radius of convergence. More precisely, if $f(x) = \sum_{n=0}^\infty a_n(x-a)^n$ for $|x-a| < R$, then $f(x)$ is automatically differentiable on $(a-R, a+R)$ and we have $f'(x) = \sum_{n=1}^\infty n a_n (x-a)^{n-1}, \int_a^x f(t)\,dt = \sum_{n=0}^\infty a_n \frac{(x-a)^{n+1}}{n+1}$ on this interval. In particular, iterating this result, we get $a_n = f^{(n)}(a)/n!$. Thus the only power series in $x-a$ that has a chance of converging to a function $f(x)$ is its Taylor series $\sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$ and no such power series can possibly converge to this function if it is not infinitely differentiable. The four most important Taylor series, all at $x=0$, are $e^x = \sum_{n=0}^\infty \frac{x^n}{n!}, \sin x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \cos x = \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!}$, and $\arctan x = \sum_{n=0}^\infty (-1)^n \frac{x^{2n+1}}{2n+1}$; the first three of these series have infinite radius of convergence while the last one has radius of convergence 1. We also have the geometric series $\sum_{n=0}^\infty ax^n$, which converges to $\frac{a}{1-x}$ if $|x| < 1$ or $a = 0$ and diverges otherwise. If a Taylor series $f(x) = \sum_{n=0}^\infty a_n(x-a)^n$ happens to converge at $x = a+R$ or $a-R$, then it automatically converges to the "right" value, that is, to $\lim_{x\to(a+R)^-}$ or $\lim_{x\to(a-R)^+} f(x)$, respectively (assuming these limits exist).

More generally, a series $\sum a_n$ with nonnegative terms a_n converges if and only if it has bounded partial sums; this holds if $a_n \le b_n$ for sufficiently large n and $\sum b_n$ is a known convergent series, while it fails if $a_n \ge b_n$ for sufficiently large n and $\sum b_n$ is a known divergent series. If the series $\sum b_n$ is such that $\frac{a_n}{b_n} \to L$ as $n \to \infty$ for some finite nonzero L, then the series $\sum a_n, \sum b_n$ converge or diverge together. A series $\sum a_n$ with nonnegative terms converges whenever $\frac{a_{n+1}}{a_n} \to L$ as $n \to \infty$ if $L < 1$ and diverges if $L > 1$. A *decreasing* sequence $\sum a_n$ with nonnegative terms such that there is a decreasing continuous function $f(x)$ with $f(n) = a_n$ converges if $\int_1^\infty f(x)\,dx$ converges and diverges otherwise. In particular the p-series $\sum_{n=1}^\infty n^{-p}$ converges if $p > 1$ and diverges otherwise. Apart from series with nonnegative terms, the main examples of convergent series we have seen are alternating series $\sum a_n$, for which $a_n = (-1)^n b_n$ and the b_n form a decreasing sequence of numbers going to 0 as n goes to ∞. Any such series converges. In particular, $\sum_{n=0}^\infty \frac{(-1)^n}{n+1}$ and $\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}$ both converge (to $\ln 2$ and $\pi/4$, respectively).

Thus we can solve certain differential equations by power series: given a linear equation like $y' + p(t)y = q(t)$ or $y'' + p(t)y' + q(t)y = r(t)$, where the functions p, q, r are assumed to have convergent Taylor series expansions, say at $x = 0$, we can inductively work out formulas for the coefficients in a power series solution $\sum a_n t^n$ of such equations. You should be able to do this in some simple cases.

Given a linear homogeneous differential equation $y'' + p(t)y' + q(t)y = 0$, there are always two particular solutions y_1, y_2 such that the general solution takes the form $c_1y_1 + c_2y_2$ for constants c_1, c_2; in fact, given *any* two solutions y_1, y_2 with neither a constant multiple of the other, the general solution takes this form. Only rarely can we find two such solutions, however (or even one nonzero solution). Two important cases where uniform formulas for such solutions exist are the constant-coefficient case $ay'' + by' + cy = 0$ and the Euler case $t^2y'' + aty' + by = 0$. In the first case, we look at the characteristic equation $ar^2 + br + c = 0$. Then e^{rt} is a solution of the differential equation whenever r is a root of the characteristic equation; more precisely, if r_1, r_2 are the distinct real roots of the characteristic equation, then we may take $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}$; if there is just one repeated real root r, then we may take $y_1 = e^{rt}, y_2 = te^{rt}$; if there are two conjugate nonreal complex roots $\alpha \pm \beta i$, then we may take $y_1 = e^{\alpha t} \cos \beta t, y_2 = e^{\alpha t} \sin \beta t$. In particular, the 0 solution is asymptotically stable if and only if both roots have strictly negative real parts (or are negative real numbers), while the 0 solution is stable but not asymptotically stable if and only if at least one root has real part 0 and 0 is not a repeated root. In the Euler case, the equation corresponding to the characteristic equation is the indicial equation $r(r-1) + ar + b = 0$. Then t^r is a solution if and only if r satisfies this equation. Hence if there are two distinct real roots r_1, r_2, then we may take $y_1 = t^{r_1 `}, y_2 = t^{r_2}$; if there is just one repeated root r, then we may take $y_1 = t^r, y_2 = t^r \ln t$; if there are conjugate complex roots $a \pm bi$, then we may take $y_1 = t^a \cos \ln |bt|, y_2 = t^a \sin \ln |bt|$. We should take $t > 0$ throughout in this last solution, to make sure that all functions are defined.