Lecture 2-7

We now exploit the definition of the Laplace transform on certain discontinuous functions to solve nonhomogeneous differential equations with discontinuous right sides. From a purely mathematical point of view we might feel no particular motivation to study such equations, there are many situations in physics and engineering when discontinuous forcing terms arise naturally; think of turning a switch off or on, or imposing an instantaneous force on a system, say by striking it with a hammer. It is handy to have techniques available that can handle such equations in a streamlined and uniform manner.

Consider for example the equation $2y''+y'+2y = g(t)$ with conditions $y(0) = y'(0) = 0$ (Example 1 in the text, p. 336), where $g(t)$ represents a unit voltage pulse between $t = 5$ and $t = 20$, so that $g(t) = 1$ for $5 \leq t < 20$ and $g(t) = 0$ otherwise. Then $g(t)$ is the difference $u_5 - u_{20}$ of the step functions u_5, u_{20} defined earlier. Accordingly, taking the Laplace transform of our equation and writing $Y(s)$ for the transform of y, we get $2s^2Y(s) - 2sy(0) - 2y'(0) + sY(s) - y(0) + 2Y(s) = (2s^2 + 2s + 2)Y(s) = \mathcal{L}(u_5 - u_{20}) = \frac{e^{-5s}-e^{-20s}}{s}$, so $Y(s) = \frac{e^{-5s}-e^{-20s}}{s(2s^2+s+2)}$. Decomposing $\frac{1}{s(2s^2+s+2)}$ by partial fractions, we get $\frac{1/2}{s} + \frac{-s-(1/2)}{2s^2+s+2}$, whose inverse Laplace transform works out to be $h(t) = \frac{1}{2} - \frac{1}{2}[e^{-t/4}\cos(\sqrt{15}t/4) + (\sqrt{15}/15)\sin(\sqrt{15}t/4)]$. (I apologize for the ugly numbers here, for which I however bear no responsibility.) Making use of the formula for the transform of $u_c(t)f(t-c)$, we see that $y = u_5(t)h(t-5) - u_{20}(t)h(t-20)$. This solution is subject to the following caveat: it is not actually differentiable everywhere, but it has left and right first and second derivatives everywhere and at every point these derivatives satisfy the given equation if we allow ourselves to take one-sided limits (on either side) at any point where the right side is discontinuous. Thus for $t < 5$, the solution is just the 0 function; for $t > 20$ its graph exhibits damped oscillation, as we know holds for any solution to $2y'' + y' + 2y = 0$. For $5 < t < 20$ the graph is that of a constant function plus a damped oscillation; at $t = 5$ and $t = 20$ the solution is not actually differentiable, but does have left and right first and second derivatives.

A more subtle kind of irregularity can occur if the right side $g(t)$ is continuous but not differentiable. For example, consider the equation $y'' + 4y = g(t)$ with the usual conditions $y(0) = y'(0) = 0$, where $g(t) = 0$ for $0 \leq t < 5, (t-5)/5$ for $5 \leq t < 10$, and 1 for $t \geq 10$ (Example 2, p. 338 in the text). Here the function $g(t)$ is continuous for all t but fails to be differentiable at $t = 0$ and $t = 5$. We may rewrite $g(t)$ as $[u_5(t)(t-5) - u_{10}(t)(t-10)]/5$; taking Laplace transforms and writing $Y(s)$ for the Laplace transform of our solution y, we get $(s^2 + 4)Y(s) = (e^{-5s} - e^{-10s})/5s^2$. Writing $\frac{1}{s^2(s^2+4)}$ as the difference $\frac{1/4}{s^2} - \frac{1/4}{s^2+4}$, and taking inverse Laplace transforms, we get $y = [u_5(t)h(t-5) - u_{10}(t)h(t-10)]/5$, where $h(t) = \frac{1}{4}t - \frac{1}{8}\sin 2t$. In this case the solution is actually twice differentiable everywhere (as the existence-uniqueness theorem guarantees it would have to be), but fails to be *three* times differentiable at $t = 5$ and $t = 10$, corresponding to the failure of $g'(t)$ to exist at those points. Everything is fine (as usual) if we take left or right-hand limits, even at the bad points.

Often in practice we are given periodic forcing functions $g(t)$, possibly discontinuous at infinitely many points. For example a *square wave* might take the value 1 for $t \in [0, 1] \cup [2, 3] \cup \ldots$, and so on, and 0 at other values of t. Here it is handy to observe that the Laplace transform of any function $f(t)$ that is periodic with period T (so that $f(t+T) = f(t)$ for all t) satisfies $\mathcal{L}f = \frac{\int_0^T e^{-st} f(t)\, dt}{1-e^{-sT}}$

Using the formula given last time for the Laplace transform $\mathcal{L}tf$ of tf in terms of that of f, we can solve a very interesting linear differential equation to which the existence-uniqueness theorem does not apply. Consider *Bessel's equation of order zero*, which reads $ty'' + y' + ty = 0$. If initial conditions are imposed at $t = 0$, there is no guarantee in advance that this equation has a solution, since it does not allow us to solve for y'' in terms of t, y, y' in a way that is defined at $t = 0$. Let's assume however that it does have a solution y such that $y(0)$ and $y'(0)$ both exist and are finite. Taking Laplace transforms of both side of the equation and writing $Y(s)$ for the transform of y, we get $(1+s^2)Y'(s) + sY(s) = 0$, a differential equation whose general solution is easily seen to be $Y(s) = c(1+s^2)^{-1/2}$, where c is a constant. Writing $(1+s^2)^{-1/2} = s^{-1}(1+s^{-2})^{-1/2}$, expanding $(1+s^{-2})^{-1/2}$ via the binomial series, and taking inverse Laplace transforms term by term, we get $y = c\sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{2^{2n}(n!)^2} = cJ_0(t)$, where $J_0(t)$ is the *Bessel function of the first kind of order zero*. Note that the series for $J_0(t)$ has infinite radius of convergence and finite derivatives of all orders at $t = 0$; taking Laplace transforms (or by direct calculation) one checks that it indeed satisfies Bessel's equation of order zero. It turns out that the only solutions to this equation remaining bounded as $t \to 0$ are constant multiples of $J_0(t)$, so that the full conclusion of the existence-uniqueness theorem indeed fails for this equation. Nevertheless, this single function is important and useful enough to deserve a special name. Its values at any point can be computed to any desired accuracy from the series.