Lecture 2-6

For our final unit on differential equations, we turn to Laplace transforms (Chapter 6 of the text). The name "transform" indicates that these are maps from functions to functions. They are useful in solving differential equations because they convert such equations to purely algebraic ones, which can then be solved directly. Also, because they are defined in terms of integrals rather than derivatives, they are defined on many discontinuous functions, which would otherwise be quite tricky to handle with the tools we have used so far.

Given a function $f(t)$, its *Laplace transform* $\mathcal{L}f(t)$ is the function defined by the improper integral $\mathcal{L}f(s) = \int_0^\infty e^{-st} f(t)\,dt$ for $s > 0$; notice that $\mathcal{L}f(t)$ is taken to be a function of s rather than t. We often write just $\mathcal{L}f$ for $\mathcal{L}f(t)$. Of course $\mathcal{L}f(t)$ is defined only if the integral converges; but we now know that this happens for $s > a$ whenever (for example) $f(t) \le Ke^{at}$ for some constants K, a. (In general, we will not pay too much attention to the domain of $\mathcal{L}f$, as long as this domain includes all $s > a$ for some a). This condition certainly does not hold for all functions (e.g. the function $f(t) = e^{t^2}$ does not satisfy it), but it does hold for most functions arising in applications. The special case where $f(t) = t^p$ and $s = 1$ is already interesting. The integral $\int_0^\infty e^{-t} t^{p-1}\,dt$ converges exactly for $p > 0$. This integral is called the *gamma function* $\Gamma(p)$. Integration by parts shows that $\Gamma(p) = (p-1)\Gamma(p-1)$ for any integer $p \ge 1$, whence an easy induction shows that $\Gamma(p) = (p-1)!$ for any positive integer p. But since $\Gamma(x)$ is defined for any real $x > 0$, we have just *interpolated* the factorial function, that is, defined it for many non-integral values. In fact, we can go further, using the recurrence $\Gamma(t+1) = t\Gamma(t)$ to *define* $\Gamma(t)$ for all real t, *apart* from nonpositive integers. (The integral $\int_0^\infty t^{-1} e^{-t}\,dt$ diverges, so even with the recurrence relation we cannot define $\Gamma(-n)$ for any nonnegative integer n). For example, changing variables, we get $\mathcal{L}t^{-1/2} = (2/\sqrt{s}) \int_0^\infty e^{-t^2}\,dt$, which we saw last quarter equals $\sqrt{\pi/s}$. From the recurrence relation we then get $\mathcal{L}t^{1/2} = \sqrt{\pi}/(2s^{3/2})$. By a simple change of variable we get $\mathcal{L}t^n = n!/s^{n+1}$ for any nonnegative integer n. More generally we have $\mathcal{L}t^p = \Gamma(p+1)/s^{p+1}$ for any real $p > -1$. Three other simple Laplace transforms are $\mathcal{L}e^{at} = \int_0^\infty e^{(a-s)t}\,dt = \frac{1}{s-a}$ (for any $s > a$), $\mathcal{L}\sin at = \int_0^\infty e^{-st} \sin at\,dt = a/(s^2 + a^2)$, and $\mathcal{L}\cos at = s/(s^2 + a^2)$, where we take $s > 0$ in both of the last formulas; you will work out these formulas in homework next week. A table of useful Laplace transforms of particular functions is given on p. 321 of the text; note that every entry this in this table is given with a reference to a section (and in some cases a HW problem) in the book.

The usefulness of Laplace transforms as a tool for solving differential equations stems from the formula $\mathcal{L}f' = s\mathcal{L}f - f(0)$, which follows at once by integration by parts. More generally we have $\mathcal{L}f^(n) = s^n \mathcal{L}f - s^{n-1} f(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$. This last fact makes it especially convenient to use Laplace transforms to solve initial-value problems where the initial conditions are imposed at time $t = 0$. Thus in Example 1 in the text (p. 318) we have the equation $y'' - y' - 2y = 0$ coupled with the initial conditions $y(0) = 1, y'(0) = 0$. Earlier we solved equations like this by guessing that a suitable exponential function $y(t) = e^{rt}$ would solve the equation if r is suitably chosen; now we can use transforms to arrive at this solution without guesswork. Indeed, taking the transform of both sides of the equation $y'' - y' - 2y = 0$ and writing $Y(s)$ for the transform $\mathcal{L}y$, we get $s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = 0$, whence $(s^2 - s - 2)Y(s) + (1 - s) = 0, Y(s) = (s-1)/(s^2 - s - 2)$ from the initial conditions. Using the partial fraction decomposition $Y(s) = (s-1)/(s^2 - s - 2) = (1/3)/(s-2) + (2/3)/(s+1)$ and the uniqueness of any continuous function with a specified Laplace transform, we get $y = (1/3)e^{2t} + (2/3)e^{-t}$, since the transforms of e^{2t} and e^{-t} are $1/(s-2)$ and $1/(s+1)$, respectively. Not only were we led to exponential functions directly here, but we solved the initial-value problem without first having to derive the general solution to the differential equation.

One disadvantage of using Laplace transforms to solve linear differential equations is that there is no formula for the Laplace transform $\mathcal{L}fg$ of a product fg in terms of the transforms of f and g. Thus for most linear differential equations (with nonconstant coefficients) we cannot solve them directly by Laplace transforms, In the special case $f(t) = t$, however, we do have such a formula, arising from the following general fact: if we define a function $G(s)$ of one variable s by integrating a function $g(s,t)$ of two variables s, t with respect to t (treating s as a constant), then the derivative $G'(s)$ of G with respect to s is gotten by integrating the s-partial derivative g_s of $g(s,t)$ with respect to t (so that the operations of differentiating with respect to one variable and integrating with respect to the other commute with each other). Thus the transform $\mathcal{L}(-t)^n f(t)$ equals $\mathcal{L}f^{(n)}(s)$. In particular the transform $\mathcal{L}t^n e^{at}$ is given by $n!/(s-a)^{n+1}$. Using this formula we are rapidly led to the general solution of a linear differential equation with constant coefficients in the case where the characteristic equation has a repeated real root, again without the necessity of guessing that a power of t times an exponential function might solve the equation.

Laplace transforms really come into their own in solving nonhomogeneous equations $y'' + ay' + by = r(t)$ for certain *discontinuous* functions $r(t)$; note that we have not dealt with such equations so far. The simplest kind of discontinuous function is the *step function* $u_c(t)$, defined to equal 1 for $t \geq c$ and 0 for $t \leq c$. By a direct calculation we find that the transform $\mathcal{L}u_c = e^{-cs}/s$; here for the first time we see a Laplace transform that is not a rational function. More generally, by a simple change of variable, we compute that the transform $\mathcal{L}u_c f_c(t)$ is $e^{-cs}F(s)$, where F is the transform of f; here c is a positive constant, $f(t)$ is an integrable function defined for $t \geq 0$, and $f_c(t)$ is defined to be 0 for $t \in [0, c)$ and to be $f(t-c)$ for $t \geq c$ (this is my private notation; henceforth we will denote this function simply by $f(t-c)$). We will use this handy formula to solve a variety of nonhomogeneous equations with discontinuous right-hand sides later.