Lecture 2-5

We now briefly sketch the main ideas in Chapter 5, indicating how power series can be used to solve differential equations, even though the sums of the series are not usually recognizable in terms of familiar functions. One of the simplest equations not with constant coefficients is *Airy's equation*, which appears in an 1838 paper of George Airy on optics. The equation is $y'' - xy = 0$. Here, if we assume that $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution representable by power series, then on differentiating twice term by term and multiplying by x, we get $y'' = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n = xy = \sum_{n=0}^{\infty} a_n x^{n+1}$, whence $2a_2 + \sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n = \sum_{n=1}^{\infty} a_{n-1}x^n$. Equating coefficients of equal powers of x, we get $a_2 = 0, (n+2)(n+1)a_{n+2} = a_{n-1}$, for $n \geq 1$. It follows at once that $a_{2+3n} = 0$ for all n; all coefficients a_{3n} are completely determined by a_0, while all coefficients a_{3n+1} are likewise determined by a_1. More precisely, we have two independent solutions of the form $y_1 = a_0(1 + \sum_{n=1}^{\infty} \frac{x^{3n}}{3n(3n-1)(3n-3)(3n-4)\cdots(3)(2)})$ and $y_2 = a_1(x + \sum_{n=1}^{\infty} \frac{x^{3n+1}}{(3n+1)(3n)(3n-2)\cdots(4)(3)})$. These power series are not summable in terms of functions we have seen before; but they are easily seen to converge for all x and thus define new and interesting functions in their own right. We call them (not surprisingly) *Airy functions*. One can derive properties of these functions from the power series expansions alone, just as we can do the same for $e^x, \sin x, \cos x$ from their power series. Airy functions are not as nice as these last functions, to be sure; but they have some qualitative features in common. For example, for negative x, Airy functions behave like solutions to $y'' + ay$ for a a positive constant; i.e. like the trigonometric functions $\sin x$ and $\cos x$. For x positive, Airy functions behave like the solutions to $y'' - ay = 0$ for a a positive constant; i.e. like the hyperbolic functions. The oscillations of Airy functions are not constant but rather decay in amplitude and increase in frequency as the distance form the origin increases. Airy functions have been extensively studied and their properties are well known to applied mathematicians and scientists.

More generally, consider any linear differential equation $y'' + p(t)y' + q(t)y = r(t)$, where p, q, r are functions of t that are analytic at at a point $t = t_0$. Then a basic theorem asserts that *the unique solution to the initial-value problem $y'' + p(t)y' + q(t)y = r(t), y(t_0) = y_0, y'(t_0) = y_1$ is also analytic at $t = t_0$, and in fact given by a Taylor series at that point whose radius of convergence is at least the minimum of the radii of convergence of the Taylor series of $p(t), q(t), r(t)$ at $t = t_0$.* More generally, given the equation $p(t)y'' + q(t)y' + r(t)y = s(t)$ together with the initial conditions $y(t_0) = y_0, y'(t_0) = y_1$, if all of the ratios $q(t)/p(t), r(t)/p(t), s(t)/p(t)$ are analytic at $t = t_0$, with Taylor series having R as their minimum radius of convergence, then the unique solution admits at Taylor series at that same point with radius of convergence at least R.

In certain cases, even if the coefficients of y' and y in a linear equation have singularities at the point $t = t_0$ specified in initial conditions, there will still be a small modification of a Taylor series that will solve the equation. Take $t_0 = 0$ for simplicity. We have already seen that the Euler equation $t^2y'' + aty' + by = 0$ typically admits two solutions t^{r_1}, t^{r_2} that are pure powers of t, even though when this equation is divided by t^2, so that the coefficient of y'' in it is 1, the coefficients of y' and y both blow up at $t = 0$. In general, any equation of the form $t^2y'' + tp(t)y' + q(t)y = 0$ where p, q are analytic at $t = 0$ and do not vanish there is said to have $t = 0-$ as a *regular singular point*. Then a fundamental theorem asserts that there is always one solution $y_1(t)$ given by the power $|t|^{r_1}$ of $|t|$ times a power series in t (this product being called a *Frobenius series*), where r_1 is the larger of the two real roots r_1, r_2 of the indicial equation $r(r-1) + p_0 r + q_0 = 0$, where p_0, q_0 are the constant terms in the power series for p and q. There is also a second independent solution of the same form with r_1 replaced by the other root r_2 of the indicial equation, provided that the difference $r_1 - r_2$ is not a nonnegative integer. If this difference is a nonnegative integer, then there is another solution of the form $y_1(t)\ln|t| + |t|^{r_2}\sum_{n=0}^{\infty} a_n t^n$ for a suitable power series $\sum_{n=0}^{\infty} a_n t^n$. A particularly important special case is that of Bessel's equation, which we will discuss in more detail after we have learned about Laplace transforms, the topic of Chapter 6 of the Boyce-DiPrima text.