**Lecture 2-4**

Last time we discussed undamped vibrations in the presence of a periodic forcing term, governed by the equation $y'' + k^2y = \cos at$, where the two crucial cases are $a = k$ (where resonance occurs) and $a$ close but not equal to $k$ (where beats occur). The other important kind of vibration discussed in Chapter 3 is the damped free vibration with a forcing term, governed by the equation $my'' + \gamma y' + ky = 0$, where $m, \gamma$, and $k$ are positive constants. Here the characteristic equation is $mr^2 + \gamma r + k = 0$ and has roots $r_1, r_2 = \frac{-\gamma\pm\sqrt{\gamma^2-4km}}{2m}$, by the quadratic formula. Note that both roots $r_i$ have negative real part, so all solutions approach 0 as $t \to \infty$. There are three cases. If $\gamma^2 - 4mk > 0$, then the general solution is $c_1e^{r_1t} + c_2e^{r_2t}$; if $\gamma^2 - 4mk = 0$, so that there is just one root $r = -\gamma/2m$, then the general solution is $c_1e^{rt} + tc_2e^{rt}$, which does not differ qualitatively from the previous case. The really different case is the one where $\gamma^2 - 4mk < 0$; here the general solution is $e^{-\gamma t/2m}(c_1 \cos \mu t + c_2 \sin \mu t$, where $\mu = \frac{\sqrt{4km-\gamma^2}}{2m} > 0$. In this case alone we get oscillatory solutions; they are not periodic, but their graphs resemble a cosine wave whose amplitude decreases as $t$ increases. The parameter $\mu$ determines the frequency with which the mass oscillates and is accordingly called the *quasi frequency*. If we compare $\mu$ to the frequency $\omega_0$ that the mass would have if damping were not present (i.e. if $\gamma$ were 0), we find that $\mu/\omega_0 = \sqrt{1 - \frac{\gamma^2}{4km}}$, so that the damping reduces the frequency of the oscillation. The quantity $T_0 = 2\pi/\mu$ is the time between successive maxima or successive minima of the position and is called the *quasi period*.

Now we give a general method for solving any nonhomogeneous equation $y''+ay'+by = r(t)$, where $r(t)$ is any continuous function; here we will find that there is a universal formula for a particular solution, based on knowledge of the functions $y_1, y_2$ spanning the solution space to the homogeneous equation $y'' + ay' + by = 0$; this formula has very much the flavor of the formula for the general solution to the first-order linear nonhomogeneous equation. The basic idea is to start with the solution to the homogeneous equation, namely $c_1y_1 + c_2y_2$, and replace the constants $c_i$ by functions $v_i$ (whence the name "variation of parameters"). Setting $y = v_1y_1 + v_2y_2$, we find that $y' = v_1'y_1 + v_1y_1' + v_2'y_2 + v_2y_2'$. This rather awkward sum of four terms becomes simpler if we assume the first and third terms sum to 0, so let us assume this, i.e. that $v_1'y_1 + v_2'y_2 = 0$. Continuing, we get $y'' = v_1''y_1 + v_1y_1'' + v_2''y_2 + v_2y_2''$. Computing $y'' + ay' + by$, we get $v_1'y_1' + v_2'y_2'$ (the coefficients of $y_1$ and $y_2$ cancel), so the conditions on $v_1, v_2$ for $v_1y_1 + v_2y_2$ to be a solution are that $v_1'y_1 + v_2'y_2 = 0, v_1'y_1' + v_2'y_2' = r(t)$. This is a linear system of equations in two unknown functions $v_1', v_2'$ whose unique solution is easily computed to be $v_1' = -ry_2/W, v_2 = ry_1/W$, where $W = W(y_1, y_2) = y_1y_2' - y_2y_1'$ is the Wronskian of $y_1$ and $y_2$, defined earlier. Integrating, we find that our particular solution to $y'' + ay' + by = 0$ is given by $(\int(-ry_2/W)\,dt)y_1 + (\int(ry_1/W)\,dt)y_2$. This solution applies whenever a spanning set $y_1, y_2$ for the solution space to any homogeneous differential equation $y'' + py' + qy = 0$ is known, to give a solution to the nonhomogeneous equation $y'' + py' + qy = r$, even if the coefficients $p, q$ are not constant.

As a nice example, consider first the Euler equation $t^2y'' + 4ty' + 2y = 0$. This equation does not have constant coefficients, but if we try a solution of the form $x^r$, we find that the left side is a multiple of $x^r$, so if we choose $r$ properly the left side will be 0. The condition on $r$ is that $r(r-1)+4r+2=(r+1)(r+2)=0$, so a spanning set of solutions to this equation is given by $y_1=t^{-1}$ and $y_2=t^{-2}$. Now look at the nonhomogeneous equation $t^2y''+4ty'+2y=t^{-1}e^t$. Computing the Wronskian $W(t^{-1},t^{-2})$ we get $(t^{-1})(-2t^{-3})-(t^{-2})(-t^{-2})=-t^{-4}$. Then $v_1'=-ry_2/W=te^t, v_1=(t-1)e^t$; similarly $v_2'=ry_1/W=-t^2e^t, v_2=(-t^2+2t-2)e^t$. Hence the general solution to $t^2y''+4ty'+2y=t^{-1}e^t$ is $(t-1)e^tt^{-1}+(t^2-2t+2)e^t(t_2)+c_1t^{-1}+c_2t^{-2}=e^t(t^{-1}-2t^{-2})+c_1t^{-1}+c_2t^{-2}$. The particular solution $e^t(t^{-1}-2t^{-2})$ is just barely simple enough that one could imagine working it out by undetermined coefficients, but I think we can all be grateful for the variation of parameters formula. Another example, this time with constant coefficients, occurs in the book on p. 187; it is the equation $y''+4y=3\csc t$. Here $\csc t$ and its derivatives involve infinitely many different functions, so one could not begin to apply the method of undetermined coefficients. Plugging in the above formulas for $v_1', v_2'$, though, we get $v_1'=-3\cos t, v_1=-3\sin t$, while $v_2'=(3/2)\csc t-3\sin t, v_2=(3/2)\ln|\csc t-\cot t|+3\cos t$. Hence the general solution is $-3\sin t\cos 2t+(3/2)\ln|\csc t-\cot t|\sin 2t+3\cos t\sin 2t+c_1\cos 2t+c_2\sin 2t$, which we may rewrite by the trigonometric addition formulas as $3\sin t+(3/2)\ln|\csc t-\cot t|\sin 2t+c_1\cos 2t+c_2\sin 2t$. Of course, in general we may not be able to write down the integral called for in the general solution to the nonhomogeneous equation, even if we know the solution to the homogeneous equation, but at least we have a general expression for this solution which moreover makes explicit how this solution depends on the right side $r(t)$. For example, if the Wronskian $W$ is small, then this solution will be unstable, changing markedly if $r(t)$ changes by only a little; but if $W$ is large, then the solution will change by only a little if $r(t)$ changes by a little.