Lecture 2-3

Continuing with differential equations, we consider now linear *nonhomogeneous* equations $y'' + py' + qy = g(t)$, where p, q are functions of t (though we will soon specialize down to the case where $p = a$ and $q = b$ are constants). We observe first that for any two solutions y_1, y_2, the difference $y_1 - y_2$ is a solution of the homogeneous equation $y'' + py' + qy = 0$, and conversely if y_1 solves the nonhomogeneous equation and $y_1 - y_2$ solves the homogeneous equation, then y_2 also solves the nonhomogeneous equation. So it is enough to find just one solution to $y'' + py' + qy = g(t)$ whenever we can completely solve the homogeneous equation $y'' + py' + qy = 0$. In many cases we can guess the general shape of a particular solution, which will involve one or more undetermined constants, and then work out what those constants must be to solve the equation. For example, given the equation $y'' - 3y' - 4y = 3e^{2t}$ (see p. 177 of the text), it is not too much of a stretch to assume that we should be able to find a solution of the form Ae^{2t} for some constant A; plugging Ae^{2t} into the equation, we get $(4A - 6A - 4A)e^{2t} = 3e^{2t}$, whence the choice $A = -1/2$ solves the equation. A slightly more complicated example (p. 178) is the equation $y'' - 3y' - 4y = 2\sin t$; here we might at first look for a solution of the form $A\sin t$, but since the derivative of the sine is the cosine, we are rapidly led to broaden our search for solutions, considering functions of the form $A\sin t + B\cos t$ for some constants A and B. Equating coefficients of $\sin t$ and $\cos t$ on both sides of the equation, we get the linear system of equations $-5A+3B = 2, -3A-5B = 0$, whose solution is $A = -5/17, B = 3/17$. Hence one solution of our equation is $(-5/17)\sin t + (3/17)\cos t$ and the general solution is $(-5/17)\sin t + (3/17)\cos t + c_1e^{4t} + c_2e^{-t}$.

Things get even a little more complicated if the right-hand side $g(t)$ happens to be a solution to the homogeneous equation. For example (p. 180) take the equation $y'' - 3y' - 4y = 2e^{-t}$. We have already noted above that e^{-t} solves the homogeneous equation, so it is no use to look for solutions to the above equation of the form Ae^{-t}. Instead (taking our cue from the solution to homogeneous equation whose characteristic equation has repeated roots) we look for a solution of the form Ate^{-t}; that is, we multiply the homogeneous solution by a constant time t. Plugging in this function, we get $(-2A - 3A)e^{-t} = 2e^{-t}$ and the particular solution is $(-2/5)te^{-t}$.

In general, if the right side $g(t)$ of the equation $y'' + ay' + by = g(t)$ is a polynomial of degree d, then a particular solution can be found which is a polynomial of degree d; if the right side $g(t)$ is a polynomial of degree d times an exponential e^{rt}, then a particular solution exists that is a polynomial of degree d times e^{rt}, or of this form times t if e^{rt} is a solution to the homogeneous equation. If $g(t)$ is a a polynomial of degree d times a combination of $\sin at$ and $\cos at$, then again a particular solution can be found of the same form, multiplied by t if $\sin at$ and $\cos at$ satisfy the homogeneous equation.

As a physical application, consider the motion of a vibrating string on which an external periodic vibration is imposed. If the string vibrates without damping, the left side of the differential equation governing its motion takes the form $mu'' + ku$, where m, k are positive constants. If no external vibration is imposed, then the general solution is $c_1 \cos\sqrt{k/m}t + c_2 \sin\sqrt{k/m}t$, as we have already seen; this periodic function describes what is called *simple harmonic motion*. Now suppose that an external periodic vibration $F_0 \cos\omega t$ is imposed, so that the differential equation becomes $mu'' + ku = F_0 \cos\omega t$. Assume first that $\omega \neq \omega_0 = \sqrt{k/m}$. In this case it is not difficult to see that a particular solution takes the form $A \cos \omega t$ for some constant A; if we assume that the system is initially at rest, with $u(0) = u'(0) = 0$, then the solution turns out to be a combination of $\cos \omega t$ and $\cos \omega_0 t$, which may be rewritten as a constant times the product of $\sin \frac{(\omega_0 - \omega)t}{2}$ and $\sin \frac{(\omega_0 + \omega)t}{2}$. If $|\omega_0 - \omega|$ is small, then $\sin \frac{(\omega_0 + \omega)t}{2}$ is rapidly oscillating compared to $\sin \frac{(\omega_0 - \omega)t}{2}$. The physical result is a rapidly oscillating function with slowly varying amplitude; if the string vibrates in air at a frequency such that we can hear the vibrations, then we get a "wa-wa" effect that is called a *beat*. Now assume that $\omega_0 = \omega$. In this case we have seen that we must multiply our purely trigonometric functions by t; the solution $u(t)$ starting from rest is a combination of $\sin \omega_0 t$ and $t \cos \omega_0 t$. Note that the qualitative behavior has completely changed from what it was in the homogeneous case; the oscillation now increase without bound as $t \to \infty$. This phenomenon is called *resonance*; it explains why when you are singing in the shower you often find that notes at certain pitches sound much louder than others. These notes correspond to natural frequencies of materials in the shower; if you sing at these frequencies your volume (or amplitude) increases over time rather than decreasing or remaining constant.