Lecture 2-26

We now review the material we have learned in vector algebra. The *vector space* $\mathbb{R}^n$ consists of all n-tuples $(r_1, \ldots, r_n)$, with the $r_i \in \mathbb{R}$. The sum $\vec{a} + \vec{b}$ of two vectors $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ is $(a_1 + b_1, \ldots, a_n + b_n)$; if $c \in \mathbb{R}$, then the scalar multiple $c\vec{a} = \vec{a}c = (c(a_1, \ldots, ca_n)$. These two formulas give the bare bones of geometry in $\mathbb{R}^n$; they are fleshed out by the further formula $||\vec{a}|| = \sqrt{a_1^2 + \ldots + a_n^2}$ for the length of the vector $\vec{a}$ and the formula $\sum_{i=1}^n a_i b_i$ for the dot product $\vec{a} \cdot \vec{b}$ of $\vec{a}$ and $\vec{b}$. We use the dot product to define the angle θ between $\vec{a}$ and $\vec{b}$ (if both $\vec{a}$ and $\vec{b}$ are nonzero), taking $\cos\theta = \frac{\vec{a} \cdot \vec{b}}{||\vec{a}||||\vec{b}||}$; this generalizes our earlier assertion that $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$. The formula for $\cos\theta$ makes sense since its numerator is known to have absolute value at most that of the denominator (the Cauchy-Schwarz inequality); as usual when taking arccosines, we take $0 \leq \theta \leq \pi$ by definition. In particular, for a fixed nonzero vector $\vec{a}$, the unique unit vector $\vec{u}$ maximizing $\vec{a} \cdot \vec{u}$ is $\vec{u} = \vec{a}/||\vec{a}||$, while the unique unit vector $\vec{u}$ minimizing $\vec{a} \cdot \vec{u}$ is $-\vec{a}/||\vec{a}||$. If $\vec{b} \neq \vec{0}$, then the projection of $\vec{a}$ onto $\vec{b}$ is given by $\frac{\vec{a} \cdot \vec{b}}{||\vec{b}||}\vec{b}$; if this vector is subtracted from $\vec{a}$, the resulting vector is orthogonal to $\vec{b}$.

If $n = 3$, then we frequently find ourselves wanting to write down a vector orthogonal to two other given ones in $\mathbb{R}^n$; fortunately we have a uniform construction $\vec{a} \times \vec{b}$ of such a vector, called the *cross product* of $\vec{a}$ and $\vec{b}$ and defined by the formula $\vec{a} \times \vec{b} = (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$ if $\vec{a} = (a_1, a_2, a_3), \vec{b} = (b_1, b_2, b_3)$. This construction makes sense only in $\mathbb{R}^3$. The length $||\vec{a} \times \vec{b}||$ of the cross product $\vec{a} \times \vec{b}$ is the product $||\vec{a}||||\vec{b}|| \sin\theta$ of the lengths of $\vec{a}, \vec{b}$ and the sine of the angle between them; this cross product, in addition to being orthogonal to both $\vec{a}$ and $\vec{b}$, has direction given by the right-hand rule: if the fingers of the right hand curl from $\vec{a}$ to $\vec{b}$ along the smallest possible angle, then the right thumb points in the direction of $\vec{a} \times \vec{b}$.

We use vectors to either parametrize or give an equation for two of the basic objects of geometry, namely lines and planes. Any line L in $\mathbb{R}^n$ is specified by the choice of one point $\vec{p}$ on it together with a direction vector $\vec{v}$ for it; then L consists exactly of the vectors of the form $\vec{p} + t\vec{v}$ for some $t \in \mathbb{R}$. Of course neither $\vec{p}$ nor $\vec{v}$ is uniquely determined by L; in fact $\vec{p}$ can be any point on $\vec{L}$, while $\vec{v}$ can be the difference between any two distinct points of L. Thus the vector $\vec{v}$ cannot be $\vec{0}$ and can be replaced by any nonzero multiple of itself, while $\vec{p}$ can in turn be replaced by $\vec{p} + t_0\vec{v}$ for any $t_0 \in \mathbb{R}$. We can find the point of intersection of two lines $L_1 = \{(p_1 + tv_1, \ldots, p_n + tv_n)\}$ and $L_2 = \{(q_1 + sw_1, \ldots, q_n + sw_n)\}$ by solving the simultaneous equations $p_i + tv_i = q_i + sw_i$ for all i. As there are only two variables in this system of n equations, we see that a random pair of lines L_1, L_2 seldom intersects, whether or not L_1 and L_2 are parallel, unless the L_i both lie in $\mathbb{R}^2$.

Turning now to planes P, we have the choice of parametrizing P (in general) or of giving the equation of P. For the midterm you can confine attention to planes living in $\mathbb{R}^3$. To parametrize P, we start by choosing a point $\vec{p}$ on it and then letting $\vec{v}_1, \vec{v}_2$ be the differences between $\vec{p}$ and two other points $\vec{q}, \vec{r}$ on P, chosen so that the $\vec{v}_i$ are independent in the usual sense that neither is a multiple of the other. Then $P = \{\vec{p} + s\vec{v}_1 + t\vec{v}_2 : s, t \in \mathbb{R}\}$, so that two parameters s, t suffice to specify any point on P uniquely. If P lives in $\mathbb{R}^3$, then we can also define P by a single linear equation $\vec{n} \cdot (x, y, z) = c$, so that a point (x, y, z)

lies in P if and only if x, y, z satisfy this equation; here c and the coordinates of $\vec{n}$ are constants with these coordinates not all 0. (The vector $\vec{n}$ is called the *normal vector* of P; as with direction vectors of lines, it cannot be $\vec{0}$ and can be replaced by any nonzero multiple of itself.) This follows since given $\vec{p}, \vec{v}_1, \vec{v}_2$ as above, we can take the cross product of $\vec{v}_1 \times \vec{v}_2$ to produce a nonzero vector $\vec{n}$ orthogonal to both of them, and then any point of the form $\vec{p} + s\vec{v}_2 + t\vec{v}_2$ will have the same dot product with $\vec{n}$ that $\vec{p}$ does, so that the set of all points having this dot product with $\vec{n}$ includes P and so coincides with it. Then we can find the point of intersection between P and any line not parallel to it, which will be unique since it will be the solution to three linear equations in three unknowns. The angle between nonparallel planes P_1, P_2 is then the angle between their (nonproportional) normal vectors; the angle between a line and a plane is $\pi/2$ minus the acute angle between the line and a normal vector to the plane, this acute angle being computed by the above formula for the cosine of the angle between two nonzero vectors, replacing the dot product in the numerator by its absolute value, so that the cosine is positive and the angle lies in the first quadrant.

The derivative $\vec{r}'(t)$ of a vector-valued function $\vec{r}(t) = (r_1(t), \ldots, r_n(t))$ of one variable t is obtained by differentiating each component: $\vec{r}'(t) = (r_1'(t), \ldots, r_n'(t))$. The *unit* tangent vector $\vec{T}(t)$ is obtained from $\vec{r}'(t)$ by dividing this vector by its length (assuming $\vec{r}'(t) \neq \vec{0}$); then the unit vector $\vec{N}(t) = \vec{T}'(t)/||\vec{T}'(t)||$ in the direction of $\vec{T}'(t)$, assuming $\vec{T}'(t) \neq \vec{0}$, is called the (principal) normal vector. If $n = 3$, the unique plane containing $\vec{r}(t_0)$ and the vectors $\vec{T}(t_0), \vec{N}(t_0)$, translated so that their tails are at $\vec{r}(t_0)$, is called the osculating plane of the curve $\vec{r}(t)$ at $t = t_0$.