Lecture 2-25

We review the material on differential equations for Friday's midterm. We begin by recalling *variation of parameters*: given any linear nonhomogeneous differential equation $y'' + py' + qy = r$ for which two independent solutions y_1, y_2 to the homogeneous equations $y'' + py' + qy = 0$ are known, there is an explicit integral formula for the general solutions, namely $y = v_1 y_1 + v_2 y_2$, where $v_1 = \int(-ry_2/W)\,dt, v_2 = \int(ry_1/W)\,dt$, where $W = y_1 y_2' - y_2 y_1'$ is the Wronskian of y_1, y_2 and the integrals are indefinite ones, including the arbitrary constants; this way we really capture the general solution and it includes the general solution to the homogeneous equation, as we knew it would have to do. It is worth emphasizing that this solution works only if the coefficient of y'' in the equation is 1, as given above; if this is not the case then one must divide by this coefficient at the outset (and then worry about points where one or more denominators are 0). In the case of constant coefficients, we later saw a second solution to this equation in terms of integrals involving the right side, which I will recall later. In the method of undetermined coefficients, we guess the general form of a solution, as a combination of derivatives of $g(t)$, possibly times a power of t or $\ln t$, and then adjust the constants to satisfy the equation.

We now recall the material on Laplace transforms. The Laplace transform $\mathcal{L}f$ of a function $y = f(t)$ is given by the integral $Y(s) = \int_0^\infty e^{-st} f(t)\,dt$. The main reason that Laplace transforms are important for our purposes comes from the formula $\mathcal{L}f^{(n)} = s^n \mathcal{L}f(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - f^{(n-1)}(0)$, as this formula converts the left side of any linear constant-coefficient differential equation in y to an algebraic equation in its Laplace transform Y, which can be easily solved. By inverting the Laplace transform, we then solve for y itself. (This works well if initial conditions on y are given at $t = 0$; if they are given at some other point $t = t_0$, then we should change the lower limit of the integral in the definition of the Laplace transform to t_0, so that the above formula for the transform of the derivatives of y are expressed in terms of the value of it and its derivatives at t_0 instead).

In order to carry this out to solve actual differential equations, it is necessary to use *partial fractions* to decompose rational functions $p(s)/q(s)$ as sums of constants over linear factors and linear functions over quadratic factors. In more detail, if the denominator $q(s)$ of $p(s)/q(s)$ factors as $(s - a_1)\ldots(s - a_r)q_1(s)\ldots q_m(s)$, where the q_i are irreducible and quadratic, then we look for constants $c_1, \ldots, c_r, d_1, e_1, \ldots d_m, e_m$ such that $p(s)/q(s) = c_1/(s - a_1) + \ldots + c_r/(s - a_r) + (d_1 s + e_1)/q_1(s) + \ldots + (d_m s + e_m)/q_m(s)$. Bringing the sum of fractions to a common denominator, multiplying out and equating coefficients of all powers of s on both sides, we work out what the constants c_i, d_i, e_i are. Then by using that the inverse transforms of $b/((s - a)^2 + b^2), (s - a)/((s - a)^2 + b^2)$ are given by $e^{as} \sin bs, e^{as} \cos bs$, respectively, we compute the solution to the given equation satisfying the given conditions.

Laplace transforms, unlike the other methods we have discussed for solving differential equations, can accommodate discontinuous functions on the right side of equations. We recall that the transform of the *unit step function* $u_c(t)$ (defined to be 1 for $t \geq c$ and 0 for $t < c$) is e^{-cs}/s; more generally, the transform of $u_c(t)f(t - c)$ (the function whose graph is obtained from the graph of $f(t)$ by shifting it c units to the right) is $e^{-cs}F(s)$, the product of e^{-cs} and the Laplace transform F of f. In applying this formula it is crucial to make sure that the constant c subtracted from the variable t in $f(t-c)$ matches the constant in the step function $u_c(t)$; if this is not the case, then you must adjust the function f by subtracting another constant from its argument to make these constants match up. Given a function $r(t)$ defined to be 0 outside of an interval $[a, b]$, we can always write it as $u_a f(t-a) - u_b f(t-b)$ for a suitable function f and then apply the above formulas to compute its Laplace transform. This is the approach used to solve equations with "ramped" forcing functions (right sides which are equal to 0 until a certain point, then increase linearly from 0 to some value a at some point, then remain equal to a past that point).

Finally, we recall that the *inverse* Laplace transform of a product $F(s)G(s)$ is the *convolution* $h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$ of the inverse Laplace transforms f, g of F, G, respectively. We use this result to derive another general formula for the solution to an in homogeneous differential equation with constant coefficients (as promised above). Given the equation $ay''+by'+cy = g(t)$ with the conditions $y(0) = y_0, y'(0) = y_1$, we take the Laplace transform of both sides to get the equation $(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_1 = G(s)$ for the transform $Y(s)$ of y, where G is the transform of $g(t)$. Solving for $Y(s)$, we get $\frac{(as+b)y_0+ay_1}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$. Then the solution y is the sum of the inverse transform of $\frac{(as+b)y_0+ay_1}{as^2+bs+c}$, call it $\phi(t)$, and the inverse transform of $\frac{G(s)}{as^2+bs+c}$, call it $\psi(t)$. Then we can recognize $\phi(t)$ as the solution to the homogeneous equation $ay''+by'+cy = 0$ with the same initial conditions $y(0) = y_0, y'(0) = y_1$, while $\psi(t)$ is the convolution $\int_0^t h(t-\tau)g(\tau)\,d\tau$ of the solution $h(t)$ to $ay'' + by' + cy = \delta(t), y(0) = y'(0) = 0$ and $g(t)$. Here $\delta(t)$ is the Dirac delta function, with Laplace transform equal to 1; then this formula says in words that $\psi(t)$ is the convolution of the *impulse response* ($h(t)$ and the *forcing function* $g(t)$. Note that the formula for $\psi(t)$ is entirely independent of the solution to the homogeneous equation and so is genuinely different from the integral formula that results from variation of parameters.