Lecture 2-24

We now apply the techniques of vector calculus to study the motion of objects orbiting the sun (including both planets and comets). Setting up a coordinate system in space with the sun at the origin, the fundamental constraint on the motion of such an object is that its acceleration vector must be a negative multiple of its position vector $\vec{f}(t)$ by Newton's laws, since the force vector acting on the object equals its mass times the acceleration vector and the force vector points toward the sum. Our first claim is that the motion then necessarily takes place in a plane. To see this, look at the cross product $\vec{L}(t) = \vec{f}(t) \times m\vec{v}(t)$, where $\vec{v}(t)$ is the velocity vector and m is the mass of the object. Taking the derivative of $\vec{L}(t)$ and using the above constraint on the acceleration vector, we find that $\vec{L}(t)$ is constant and nonzero, since the object is not moving toward the sun. (In physical terms, we say that the torque of the orbiting body is $\vec{0}$ and its angular momentum is constant.) Hence the motion indeed takes place in the plane with $\vec{L}(t)$ as its normal vector, which we take to be the xy-plane. Write $\vec{f}(t)$ in polar rather than Cartesian coordinates, so that $\vec{f}(t) = (r\cos\theta, r\sin\theta)$, where both r and θ are functions of t. (Here we do not bother to parametrize by arclength.) Write $\vec{p} = (\cos\theta, \sin\theta)$, the unit vector pointing directly away from the origin and $\vec{n} = (-\sin\theta, \cos\theta)$, a unit vector orthogonal to $\vec{p}$. Note that $\vec{p}' = -\vec{n}\theta'$, while $\vec{n}' = -\vec{p}\theta'$. Differentiating and using the chain and product rules, we get $\vec{v}(t) = \vec{f}'(t) = (r'\cos\theta - r\theta'\sin\theta, r'\sin\theta + r\theta'\cos\theta) = r'\vec{p} + r\theta'\vec{n}$. Differentiating again, we get $\vec{a}(t) = \vec{v}'(t) = r''\vec{p} + r'\theta'\vec{n} + r'\theta'\vec{n} + r\theta''\vec{n} - r(\theta')^2\vec{p} = (r'' - r(\theta')^2\vec{p} + (2r'\theta' + r\theta'')\vec{n}$. We have seen that the acceleration vector points toward the origin, whence the coefficient of $\vec{n}$ in $\vec{a}(t)$ must be 0. Thus $2r'\theta' + r\theta'' = rr'\theta' + (r^2)\theta'' = 0$. We now recall that $(r^2/2)\theta'$ represents the rate of change of the area swept out by the line joining the body to the origin, by the formula for area in polar coordinates. Hence *a body orbiting the sun has the line joining it to the sun sweeping out area at a constant rate if and only if its acceleration points towards the sun*. That lines joining the planets to the sun indeed sweep out area at a constant rate is *Kepler's second law of planetary motion*, though it in fact applies equally well to comets.

Now we will show that orbiting bodies travel along conic sections. To do this we need to review some facts from last quarter about conic sections and polar coordinates. and add a new observation. Last quarter we saw that the polar equation $r = \frac{d}{1-e\cos\theta}$ with d, e constants is equivalent to $r = d + ex$ and $x^2 + y^2 = (d + ex)^2 = d^2 + 2dex + e^2x^2$. Combining coefficients of x^2 we get $(1 - e^2)(x - \frac{de}{1-e^2})^2 + y^2 = \frac{d^2}{1-e^2}$. If $e < 1$ we may rewrite this equation as $(\frac{x-\alpha}{a})^2 + y^2/b^2 = 1$, where $\alpha = de/(1 - e^2), a = d/(1 - e^2), b = d/\sqrt{1 - e^2}$. Now recall from last quarter that an ellipse in standard position with the equation $x^2/a^2 + y^2/b^2 = 1$ with $a \geq b > 0$, has foci $(\pm\sqrt{a^2 - b^2}, 0)$ on the x-axis and eccentricity $e = \sqrt{a^2 - b^2}/a$. The effect of replacing x by $x - \alpha$ in the equation is to shift the foci α to the right, so that the foci of the ellipse with polar equation $r = \frac{d}{1-e\cos\theta}$ are at $(0, 0)$ and $(2de/(1 - e^2), 0)$; the eccentricity is e, as suggested by the notation. If the angle θ is replaced by $\theta - \theta_0$ for some constant θ_0, then the ellipse is rotated about $(0, 0)$; thus $(0, 0)$ is still a focus but the major and minor axes are no longer parallel to the coordinate ones.

If instead $e > 1$, then a very similar analysis applies and shows that this polar equation defines (both branches of) a hyperbola with one focus at the origin, though of course, a point moving according to this equation must stay on one branch of the hyperbola; it cannot jump from one branch to the other.

Finally, the “degenerate” case, actually quite common in practice, occurs when $e = 1$ in the polar equation $r = \frac{d}{1-e\cos\theta}$. Here the equation reduces to $r = 1 + x, y^2 = 2x + 1$. In this case there is a unique focus, again at $(0, 0)$, and the eccentricity is 1.

Now we can show that any orbiting object travelling in such a way that its acceleration vector always points toward the sun travels along a conic section. The differential equation governing the motion, given the formula for $a(t)$ above, is $r'' - r(\theta')^2 = -k/r^2$, where k is a constant. It is too hard to solve for both r and θ in terms of t from this equation, so we instead derive another equation relating $1/r$ to its second derivative $d^2(1/r)/d\theta$ and solving for $1/r$. We have $d(1/r)/d\theta = -r'/r^2\theta' = -r'/C$, where $C = r^2\theta'$ is constant. Differentiating again, we get $d^2(1/r)/d\theta^2 = -r''/C\theta'$; adding $d^2(1/r)/d\theta^2$ to $1/r$ we get $\frac{-r''}{C\theta'} + \frac{1}{r} = \frac{(r(\theta')^2-r'')r^2}{C^2} = k/C^2$, another constant. This equation is well known to have solution $r = d/(1 - e\cos(\theta - \theta_0))$ for a suitable constant θ_0 and constants d, e, whence the object indeed travels along a conic section with the sun at one focus (possibly rotated out of standard position); the sun is the only focus if the orbit is a parabola and is one of two foci otherwise.

Of course the planets travel in elliptical orbits; if their orbits were not periodic we would not call them planets. Comets, however, are more varied; many are periodic (such as Halley’s Comet) and so travel in ellipses, but many travel in parabolas or hyperbolas. It is in an interesting empirical fact that the elliptical orbits traced by planets and comets exhibit a wide range of eccentricities, while the parabolic and hyperbolic orbits traced by comets all have eccentricities very close to 1.