Lecture 2-21

We now specialize down to curves $\vec{r} = \vec{r}(t)$ in $\mathbb{R}^3$, assuming first that $\vec{r}(t)$ has been parametrized by arclength, so that its tangent vector $\vec{T} = \vec{r}'$ is a unit vector. We have already observed as a consequence $\vec{T}'$ is necessarily orthogonal to $\vec{T}$. We now define the *curvature* $\kappa = \kappa(t)$ of $\vec{r}$ at time t to be the length $||\vec{T}'||$ of the acceleration vector $\vec{T}'$, while the unit vector $\vec{T}'/||\vec{T}'||$ is called, as mentioned last time, the *(principal) normal vector* (assuming $\vec{T}'$ is nonzero) and denoted $\vec{N}$. (Note that we have already defined curvature for plane curves in the same way.) Set $\vec{B} = \vec{T} \times \vec{N}$; then $\vec{B}$ is called the *binormal*. Then for any t with $\vec{T}'(t) \neq \vec{0}$ we have that $\vec{T} = \vec{T}(t), \vec{N} = \vec{N}(t)$, and $\vec{B} = \vec{B}(t)$ are three mutually orthogonal unit vectors; a basic fact from the linear algebra of $\mathbb{R}^3$ (which we will later generalize and prove for any $\mathbb{R}^n$) is that *any* $\vec{v} \in \mathbb{R}^3$ *is a unique linear combination* $v_{\vec{T}}\vec{T} + v_{\vec{N}}\vec{N} + v_{\vec{B}}\vec{B}$*, where for* $\vec{X} = \vec{T}, \vec{N}$*, or* $\vec{B}$ *we have* $v_X = \vec{v} \cdot \vec{X}$. For now, we can make the result more plausible by noting that by problem 13.4.40 of this week's homework, the triple cross product $(\vec{a} \times \vec{b}) \times \vec{c}$ of any three mutually orthogonal vectors $\vec{a}, \vec{b}, \vec{c}$ in $\mathbb{R}^3$ is $\vec{0}$, so that $\vec{a} \times \vec{b}$ is necessarily a multiple of $\vec{c}$.

On differentiating the identities $\vec{T}\cdot\vec{T} = \vec{N}\cdot\vec{N} = \vec{B}\cdot\vec{B} = 1$ and $\vec{T}\cdot\vec{N} = \vec{T}\cdot\vec{B} = \vec{N}\cdot\vec{B} = 0$ with respect to t we find that $\vec{X}'$ is orthogonal to $\vec{X}$ for $\vec{X} = \vec{T}, \vec{N}$, or $\vec{B}$, while the coefficient κ of $\vec{N}$ in $\vec{T}'$ is the negative of the coefficient $-\kappa$ of $\vec{T}$ in $\vec{N}'$, and similarly for $\vec{T}', \vec{B}'$ and $\vec{B}', \vec{N}'$. Thus we may write $\vec{T}' = \kappa\vec{N}, \vec{N}' = -\kappa\vec{T} + \tau\vec{B}, \vec{B}' = -\tau\vec{N}$ for some $\tau = \tau(t)$, called the *torsion* of the curve $\vec{r}(t)$ at t. (A formula in the text on p. 733 asserts that $\vec{N}' = -\kappa\vec{T} - \tau\vec{B}$, but it is more common to define τ by the formula for $\vec{N}'$ given above. Unlike κ, τ can be either positive or negative.) Just as $\kappa(t)$ measures the amount by which the curve bends per unit of length at time t, so $\tau(t)$ measures the amount by which the curve twists out of its osculating plane at the same time. The formulas for $\vec{T}', \vec{N}'$, and $\vec{B}'$ given above are called the *Frenet formulas*; they express the rates of change of the three vectors $\vec{T}, \vec{N}$, and $\vec{B}$ fundamental to the intrinsic geometry of a curve in $\mathbb{R}^3$ to the vectors themselves.

For example, returning to our old friend the circular helix $\vec{r} = (a\cos t, a\sin t, bt)$, we have $\vec{T} = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b)$, while $\vec{N} = (-\cos t, -\sin t, 0)$. The curvature κ, being the length of $d\vec{T}/ds$ (and the arclength s being equal to $\sqrt{a^2+b^2}t$ in this situation), is constantly equal to $a/(a^2+b^2)$. The binormal $\vec{B} = \vec{T}\times\vec{N}$ is then $\frac{1}{\sqrt{a^2+b^2}}(b\sin t, -b\cos t, a)$. Equating $\vec{B}'$ to $-\tau\vec{N}$, we get $\tau = b/(a^2+b^2)$, another constant. The circular helix both bends and twists at a constant rate.

What about the vast majority of curves that are not parametrized by arclength? Using the chain rule, we can work out (slightly more complicated) formulas for curvature and torsion. Thus let $\vec{r}(t)$ be any parametrized curve and write s for the arclength of this curve (measured in a fixed direction and starting at a fixed point). Write $\vec{v}(t), \vec{a}(t)$, respectively, for the velocity (or tangent) and acceleration vectors to this curve. By the formula for ds/dt, we have $\vec{v} = (ds/dt)\vec{T}$, where as usual $\vec{T}$ is the unit tangent vector. Differentiating $\vec{v}$ with respect to t and using the chain rule, we get $\vec{a} = (d^2s/dt^2)\vec{T} + (ds/dt)ds/dt)(d\vec{T}/ds) = (d^2s/dt^2)\vec{T} + \kappa(ds/dt)^2\vec{N}$, where $\vec{N}$ is the normal vector, since by definition $\kappa = ||d\vec{T}/ds||$. The last equation can also be written as $\vec{a} = v'\vec{T} + \kappa v^2\vec{N}$, where $v = ds/dt$ is the speed;

we call the coefficients $v', \kappa v^2$ the *tangential* and *normal components of acceleration*. The appearance of v^2 in the formula for the normal component explains why you feel the strongest force pulling you toward the center of the track if you race around a circular track at high speed; in fact there is no tangential component of acceleration at all if the speed is constant. Solving for κ in terms of $\vec{v}, \vec{a}$, and $v = ds/dt$ alone, we get $\kappa = \frac{||\vec{v}\times\vec{a}||}{v^3}$. This formula agrees with the earlier one derived for curves in $\mathbb{R}^2$. Similarly we have $\tau = (\vec{v} \times \vec{a}) \cdot \vec{a}'/||\vec{v} \times \vec{a}||^2$. For example, for the curve $\vec{r}(t) = (3t - t^3, 3t^2, 3t + t^3)$ we can compute that $\kappa = \tau = \frac{1}{3(1+t^2)^2}$. It usually happens (as in this case) that if the curvature or torsion is not constant, then it goes to 0 as $t \to \infty$.