## Lecture 2-20

Continuing with parametrized curves $\vec{r} = \vec{r}(t)$, regarded as differentiable vector-valued functions of a variable $t$ running over a closed interval $[a, b]$, suppose now that we are given two such curves $\vec{r}(t), \vec{s}(t)$ defined on the same interval. Then clearly $(\vec{r} + \vec{s})' = \vec{r}' + \vec{s}'$; similarly $(\alpha\vec{r})' = \alpha\vec{r}'$ for any scalar $\alpha$. More interestingly, since the dot product is bilinear (satisfying the distributive law on both sides), we can copy the proof of the product rule for differentiable functions of one variable to show that the derivative $(\vec{r} \cdot \vec{s})'$ of the real-valued function $\vec{r} \cdot \vec{s}$ is $\vec{r}' \cdot \vec{s} + \vec{r} \cdot \vec{s}'$; similarly, if $\vec{r}, \vec{s}$ take values in $\mathbb{R}^3$, then $(\vec{r} \times \vec{s})' = \vec{r}' \times \vec{s} + \vec{r} \times \vec{s}'$. If $u = u(t)$ is a differentiable function of one variable, we also have $(u\vec{r})' = u'\vec{r} + u\vec{r}'$; if in addition the composite function $\vec{r}(u(t)$ is defined on an interval, then we have the chain rule $\vec{r}(u(t))' = \vec{r}'(u(t)u'(t)$. In particular, if $\vec{r}$ is differentiable, then so is $r = ||\vec{r}(t)||$, a real valued function, except at points where $r = 0$, and we have $rr' = \vec{r} \cdot \vec{r}'$ (on differentiating the definition of $r^2$ by a dot product).

The text defines the speed of a parametrized curve in $\mathbb{R}^2$ but for some reason never explicitly generalizes this to higher dimensions. We call the length $||\vec{r}'||$ of the tangent vector of a parametrized curve $\vec{r} = \vec{r}(t)$ the *speed* of $\vec{r}$, often denoting it (as in the last paragraph) by $r = r(t)$. As in the case of curves lying in $\mathbb{R}^2$, the arclength of a curve segment $\vec{r}(t)$ as $t$ runs over $[a, b]$ is given by the integral $\int_a^b r(t)\,dt$. This length is then the least upper bound of the sums of the lengths of the line segments joining $\vec{r}(t_i)$ to $\vec{r}(t_{i+1})$ as $\{t_0, \ldots, t_n\}$ runs through all partitions of $[a, b]$ and the index $i$ runs from 0 to $n - 1$. In Example 2 in the text (p. 718), the arclength of the curve $(2\cos t, 2\sin t, t^2)$ for $0 \le t \le \pi/2$ (which follows a spiral path) is computed to be $\int_0^{\pi/2} 2\sqrt{1 + t^2}\,dt$, which by Formula 78 in the integral table in the text (found on the inside back cover) equals $[t\sqrt{1 + t^2} + \ln(t + \sqrt{1 + t^2}]_{t=0}^{t=\pi/2} = \pi/2\sqrt{1 + (\pi^2/4)} + \ln[(\pi/2) + \sqrt{1 + (\pi^2/4)}]$, or about 4.158.

A curve with constant speed 1 is said to be *parametrized by arclength*, generalizing the case where this curve lies in $\mathbb{R}^2$. As in that case, we see that any curve can in theory be reparametrized by arclength, though we may not have an explicit expression for the arclength parameter. Thanks to the chain rule, we can predict in advance what the tangent vector to a curve will be at any given point after the curve is reparametrized by arclength, assuming that the original tangent vector at the point is nonzero; it will be just the unit vector in the direction of the original tangent vector. There are just two ways to parametrize any curve by arclength, once an initial point of the curve is specified, corresponding to the two directions in which one can trace the curve starting from this point. The direction of the tangent vector to a curve at any point, if this vector is nonzero, specifies the direction in which the curve is being traced at that point. If a curve is given by $\vec{r}(t)$, starting from $\vec{r}(a)$ and moving to $\vec{r}(b)$, then the same curve, starting from its original endpoint $\vec{r}(b)$ and moving in the opposite direction to $\vec{r}(a)$, is parametrized by $\vec{r}(a + b - t)$.

If a curve $\vec{r}$ has constant speed (not necessarily equal to 1) then we have $\vec{r} \cdot \vec{r}'$, so that the position vector $\vec{r}$ of such a curve is always orthogonal to its velocity (or tangent) vector. We will use this fact repeatedly later on. An example of a curve with constant speed that we have not seen before is the *(circular) helix* $\vec{r}(t) = (a\cos t, a\sin t, bt)$ for positive constants $a, b$; here the speed is constantly equal to $\sqrt{a^2 + b^2}$. In words, this curve winds around the origin at a constant rate while also rising from the $xy$-plane at a (possibly different) constant rate. Note that the curve whose arclength was computed above is *not* a circular helix even though it traces a helical path.

If a curve $\vec{r}(t)$ is parametrized by arclength (or more generally has constant speed), so that its tangent vector $\vec{T}(t) = \vec{r}'(t)$ has constant length 1, then we have seen that the differentiated tangent vector $\vec{T}'(t)$ is always orthogonal to $\vec{T}(t)$. Whenever $\vec{T}'(t)$ is nonzero, we call the unit vector $\vec{N}(t) = \vec{T}'(t)/||\vec{T}'(t)||$ in the direction of $\vec{T}'(t)$ the *principal normal vector* and the unique plane passing through any point of the curve and containing the unit tangent and principal normal vectors at that point the *osculating plane* of the curve at that point; here "osculating" comes from the Latin word for kissing. This is the plane which most nearly captures the motion of a point tracing the curve at that point. For example, given the circular helix $(a\cos t, a\sin t, bt)$ the unit tangent vector at this point is $\vec{T}(t) = \frac{1}{\sqrt{a^2+b^2}}(-a\sin t, a\cos t, b)$ and unit normal vector is $\vec{N}(t) = (-\cos t, -\sin t, 0)$. The equation of the osculating plane is $(b\sin t)x - (b\cos t)y + az = abt$ (Example 4. p. 711, in the text).