Lecture 2-19

Now we finally return to calculus, this time in a multidimensional context. We have seen that an n-tuple $(p_1 + tv_1, \ldots, p_n + tv_n)$ of linear functions of the variable t defines a line in $\mathbb{R}^n$ (if at least one v_i is nonzero); more generally, any n functions $f_1(t), \ldots, f_n(t)$ differentiable on a closed interval $[a, b]$ define a *parametrized curve in* $\mathbb{R}^n$. For a moment, consider the more general case of n arbitrary functions $f_1(t), \ldots, f_n(t)$ defined on an interval $(a-h, a+h)$ for some $h > 0$. Since the length of any vector $\vec{v} = (v_1, \ldots, v_n) \in \mathbb{R}^n$ is bounded below by $\sqrt{n}$ times the minimum of the $|v_i|$, and above by $\sqrt{n}$ times the maximum of the $|v_i|$, it follows that $\lim_{t\to a}(f(t_1), \ldots, f(t_n))$ exists and equals the vector $\vec{a} = (a_1, \ldots, a_n)$ in the sense that given $\epsilon > 0$ there is $\delta > 0$ such that $0 < |t-a| < \delta \to ||f(t) - (a_1, \ldots, a_n)|| < \epsilon$ if and only if $\lim_{t\to a} f_i(t)$ exists and equals a_i for all i. In particular, the vector-valued function $\vec{f}(t)$ sending t to $(f_1(t), \ldots, f_n(t))$ is continuous at $t = a$ if and only if the coordinate functions $f_i(t)$ are continuous there. It also follows easily that the *derivative* $f'(a) = \lim_{t\to a} \frac{\vec{f}(t)-\vec{f}(a)}{t-a}$ exists if and only if every $f_i(t)$ is differentiable at a; if so then $f'(a) = (f_1'(a) \ldots, f_n'(a))$. (Here of course we interpret the fraction $\frac{\vec{f}(t)-\vec{f}(a)}{t-a}$ as the scalar $\frac{1}{t-a}$ times the vector $\vec{f}(t) - \vec{f}(a)$.) Thus for example the derivative of the vector-valued function (t^2, t^3, t^4) at any point t is given by $(2t, 3t^2, 4t^3)$.

Generalizing the case $n = 2$ which we have treated before, we therefore define the *tangent* or *velocity* vector of a parametrized curve $\vec{f}(t)$ at $t = t_0$ to be $\vec{f}'(t_0)$ as defined above. The *acceleration vector* at $t = t_0$ is then defined to be $\vec{f}''(t_0)$ whenever this vector is defined. Similarly, we allow ourselves to integrate n-tuples of continuous (or more generally integrable) functions coordinate by coordinate: if the coordinates of $\vec{f}(t) = (f_1(t), \ldots, f_n(t))$ are all integrable on $[a, b]$, then we define $\int_a^b \vec{f}(t)\,dt$ to be $(\int_a^b f_1(t)\,dt, \ldots, \int_a^b f_n(t)\,dt)$. A vector-valued function $\vec{f}(t)$ is then integrable on $[a, b]$ if and only if its coordinate functions are.

Thus much of the work we have done with real-valued functions of one variable carries over at once to vector-valued functions. We have to be careful, though, when it comes to theorems like the Intermediate Value Theorem asserting the existence of a number in an interval on which a function $f(t)$ takes a specified value. For example, suppose we are given a continuous vector-valued function $\vec{f}(t)$ on an interval $[a, b]$ and let $\vec{f}(a) = (a_1, \ldots, a_n), \vec{f}(b) = (b_1, \ldots, b_n)$. Then it is certainly not true in general that if $c_1, \ldots, c_n$ are any real numbers with $a_i \le c_i \le b_i$ for all i then there is $t \in [a, b]$ with $f_i(t) = c_i$ for all i; certainly for all i there is $t_i \in [a, b]$ with $f_i(t_i) = c_i$, but there is no reason to expect all of the t_i to be the same. Similarly, we cannot expect that there will be a single $t_0 \in [a, b]$ such that $f_i(t)$ is maximized (or minimized) at $t = t_0$ for all i. Finally, as we observed last quarter even in the case $n = 2$, if the f_i are all differentiable on (a, b) and continuous on $[a, b]$ there will not be a single $c \in [a, b]$ such that $f_i'(c)(b-a) = f_i(b) - f_i(a)$ for all i (think of the case $f_1(t) = \cos t, f_2(t) = \sin t, a = 0, b = 2\pi$). We do at least get a weak version of the mean-value theorem: if $||\vec{f}'(t)|| < M$ for all $t \in [a, b]$, then we must have $|f_i(x) - f_i(y)| < M|x - y|$ for all indices i and $x, y \in [a, b]$, for otherwise there would be some index i and $z \in [a, b]$ with $|f_i'(z)| \ge M$, whence $||\vec{f}'(z)|| \ge M$. We also have the following integral version of the Cauchy-Schwarz inequality: if $\vec{f}(t)$ is integrable on $[a, b]$, then $||\int_a^b \vec{f}(t)\, dt|| \leq \int_a^b ||\vec{f}(t)||\, dt$. To see this let $\vec{c} = \int_a^b \vec{f}(t)\, dt$. The result is obvious if $\vec{c} = \vec{0}$, so assume $\vec{c} \neq \vec{0}$. Then $||\vec{c}||^2 = \vec{c} \cdot \int_a^b \vec{f}(t)\, dt = \int_a^b \vec{c} \cdot \vec{f}(t)\, dt \leq \int_a^b ||\vec{c}||||\vec{f}(t)||\, dt$ by Cauchy-Schwarz. The last integral equals $||\vec{c}|| \int_a^b ||\vec{f}(t)||\, dt$ and the result follows on division by $||\vec{c}||$.

What makes vector-valued functions of a real variable so tractable is that there are only two ways to approach a real number, namely from the left or the right. Once we consider functions of a vector variable, however, matters become quite different, as there are infinitely many ways to approach even a point in the plane. For example, what is the limit of $f(x, y) = xy/(x^2 + y^2)$ the variables x, y both approach 0? At first we might be tempted to say that the limit is 0, for if one of x or y is 0, then $f(x, y) = 0$, so certainly $\lim_{y\to 0} f(0, y) = \lim_{x\to 0} f(x, 0) = 0$. On closer inspection, however, we notice that $f(x, x) = 1/2$ for any $x \neq 0$, so there is no value of $\delta > 0$ that will make $|f(x, y)| < \epsilon$ whenever $0 < ||(x, y)|| < \delta$. We are forced therefore to say that $\lim_{(x,y)\to(0,0)} f(x, y) \neq 0$; in fact, the limit does not even exist. We will return to this example later and see that it forces us to work pretty hard to define differentiability even for functions of two variables.