Lecture 2-18

We continue with $\mathbb{R}^n$, treating lines and hyperplanes in this space. Our discussion generalizes that of the text, which covers lines and (ordinary) planes in $\mathbb{R}^3$. To begin with lines, let $\vec{p}_1, \vec{p}_2$ be two distinct points in $\mathbb{R}^n$. A point on the unique lines determined by these points is then given by the sum $\vec{p}_1 + t(\vec{p}_2 - \vec{p}_1)$ of $\vec{p}_1$ and a multiple of the difference $\vec{p}_2 - \vec{p}_1$. Conversely, given any $\vec{p}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} ne \vec{0}$, the set of all points $\vec{p} + t\vec{v}$ as t runs over $\mathbb{R}$ is a line in $\mathbb{R}^n$ and all such lines arise in this way. Thus we have parametrized lines in $\mathbb{R}^n$, just as we did earlier for curves in $\mathbb{R}^2$, describing a typical point on such a line by an n-tuple $(p_1 + tv_1, \ldots, p_n + tv_n)$ of (rather simple) functions of t. We call the vector $\vec{v}$ a *direction vector* for the line; note that $\vec{v}$ can be replaced here by any nonzero multiple of itself to product another direction vector for the same line. Likewise the point $\vec{p}$ could be replaced by $\vec{p} + t_0\vec{v}$ for any $t_0 \in \mathbb{R}$. Note also that lines in $\mathbb{R}^n$ for any $n > 2$ are too complicated to have their directions specified by single numbers; such lines do not have slopes. We say that two lines with direction vectors $\vec{V}_1, \vec{v}_2$ are *parallel* if $\vec{v}_1, \vec{v}_2$ are (necessarily) nonzero multiples of each other.

In $\mathbb{R}^n$ for $n > 2$, unlike $\mathbb{R}^2$, most pairs of nonparallel lines do not intersect. For example, consider the lines L_1, L_2 passing through $(1,1,1)$ and $(0,0,0)$, respectively, and with respective direction vectors $(2,4,5)$ and $(1,2,3)$. Clearly these lines are nonparallel; if they intersected, there would be real numbers s, t such that $1 + 2s = t, 1 + 4s = 2t$, and $1 + 5s = 3t$; but already the first two of these equations are inconsistent. If two lines do intersect, then the angle between them is defined to be the angle between their direction vectors; this is independent of the choice of direction vector for both lines. Sometimes one gives equations rather than a parametrization of a line; if the coordinates d_i of a direction vector $(d_1, \ldots, d_n)$ are all nonzero, and if the line passes through the point $(p_1, \ldots, p_n)$, then one can specify this line by the equations $\frac{x_1 - p_1}{d_1} = \ldots = \frac{x_n - p_n}{d_n}$.

We turn now to hyperplanes in $\mathbb{R}^n$. Rather than parametrizing these we usually describe them by a single equation: if $(d_1, \ldots, d_n)$ is a nonzero vector and $a \in \mathbb{R}$, then the linear equation $(x_1, \ldots, x_n) \cdot (d_1, \ldots, d_n) = a$ defines a hyperplane in $\mathbb{R}^n$ and all hyperplanes arise in this manner. We call $(d_1, \ldots, d_n)$ a *normal vector* for the hyperplane; observe that such normal vectors, like direction vectors of lines, are well defined only up to a nonzero multiplicative scalar, since the equation $(x_1, \ldots, x_n) \cdot k(d_1, \ldots, d_n) = ka$ describes the same hyperplane as $(x_1, \ldots, x_n) \cdot (d_1, \ldots, d_n) = a$ if k is a nonzero number. Given a line and a hyperplane there are just three possibilities: either the line lies in the hyperplane, or it is parallel to the hyperplane, or it intersects the hyperplane in exactly one point. The line intersects the hyperplane in a point if and only if a direction vector for it is *not* orthogonal to a normal vector for the hyperplane. For example, the line L_1 defined above, passing through $(1,1,1)$ and having direction vector $(2,4,5)$, intersects the plane with equation $x + y + z = 0$ at the point $(1+2t, 1+4t, 1+5t)$, where $1+2t+1+4t+1+5t = 3+11t = 0$, so that $t = -3/11$. The point is thus $(5/11, -1/11, -4/11)$.

Any two hyperplanes in $\mathbb{R}^n$ intersect; the angle between them is defined to be the angle between their normal vectors.

A line in $\mathbb{R}^n$ is a subspace of the latter (so that, by definition, the sum of two points on it is also on it) if and only if the line passes through the origin $\vec{0} = (0, \ldots, 0)$. Similarly, a hyperplane in $\mathbb{R}^n$ is a subspace if and only if it passes through the origin, or equivalently the right-hand side of the equation defining it is 0.

For $n = 3$, it is handy to have a general construction for producing a vector $\vec{u}$ orthogonal to two given ones $\vec{v}, \vec{w}$ (so that, for instance, if one is given that plane in $\mathbb{R}^3$ contains lines with direction vectors $\vec{v}, \vec{w}$, one can work out a normal vector for the plane). Fortunately, a general construction is available: we can always take $\vec{u} = \vec{v} \times \vec{w}$, provided that the vectors $\vec{v}, \vec{w}$ are nonparallel (and if they are parallel, there will be not one but many choices for a vector orthogonal to both of them). Thus for example given the lines $L_1 = (1,1,1) + t(2,4,5)$ and $L_3 = (1,1,1) + s(2,2,3)$, the equation of the unique plane containing both of them is $2x + 4y - 4z = 2 \cdot 1 + 4 \cdot 1 - 4 \cdot 1 = 2$, since the cross product of $(2,4,5)$ and $(2,2,3)$ is $(2,4,-4)$.

Finally we give a formula that is sometimes useful for the distance from a point $\vec{P}_1$ to a line not containing it in $\mathbb{R}^3$ (given in the text on p. 677). Taking $\vec{P}_0$ to be any point on the line and $\vec{d}$ to be a direction vector for the line this distance is given by the ratio $\frac{||(\vec{P}_1 - \vec{P}_0) \times \vec{d}||}{||\vec{d}||}$, since if $\vec{Q}$ is the projection of $\vec{P}_1$ to the line, then we may take $\vec{d} = \vec{Q} - \vec{P}_0$ (assuming $\vec{P}_0 \neq \vec{Q}$) and then the required distance is $||\vec{P}_1 - \vec{P}_0|| \sin\theta$, where θ is the angle between the line from $\vec{P}_0$ to $\vec{P}_1$ and the line.