Lecture 2-12

We return to Salas-Hille for the remainder of the course this term. We begin with the basic object of study, namely the set $\mathbb{R}^n$, consisting of all n-tuples $(a_1, \ldots, a_n)$ of real numbers a_i; note that the n here is not an exponent. Note that the definition of $\mathbb{R}^n$ is essentially the same as that of $\mathbb{R}^3$ in the text, replacing (a_1, a_2, a_3) wherever it occurs by $(a_1, a_2, \ldots, a_n)$, and indeed many proofs of theorems about $\mathbb{R}^n$ have the same property, being essentially no more complicated for general n than for $n = 3$.

The first thing to say about $\mathbb{R}^n$, using the language of the first week of last quarter, is that it is an abelian group under addition, where we decree that $(v_1, \ldots, v_n) + (w_1, \ldots, w_n) = (v_1 + w_1, \ldots, v_n + w_n)$; moreover, given $(a_1, \ldots, a_n) \in \mathbb{R}^n, c \in \mathbb{R}$ there is $c(a_1, \ldots, a_n) = (ca_1, \ldots, ca_n) \in \mathbb{R}^n$, such that $c(\vec{v}+\vec{w}) = c\vec{v}+c\vec{w}, (c+d)\vec{v} = c\vec{v}+d\vec{v}, 1\vec{v} = \vec{v}, c(d\vec{v})) = (cd)\vec{v}$ for all $\vec{v}, \vec{w} \in \mathbb{R}^n, c, d \in \mathbb{R}$. We express the full set of properties here (involving vector addition, scalar multiplication, and the relationships between them) by saying that $\mathbb{R}^n$ is a *vector space* over the field $\mathbb{R}$; more generally, the set K^n of n-tuples over any field K is a vector space over K. Note that a vector here is simply an element of a vector space; vectors do not have to be n-tuples, and indeed we will later see that certain sets of functions are vector spaces as well. In $\mathbb{R}^n$ there is a notion of distance; by constructing a suitable sequence of $n-1$ right triangles and repeatedly applying the Pythagorean Theorem, we see that the distance $||\vec{v}||$ between $\vec{0} = (0, \ldots, 0)$ and $\vec{v} = (v_1, \ldots, v_n)$, also called the *norm* of $\vec{v}$, is $\sqrt{\sum v_i^2}$; similarly, the distance between $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ is $\sqrt{\sum (b_i - a_i)^2}$. We then have the *triangle inequality* for distances, which asserts for any vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ that $||\vec{v} + \vec{w}|| \leq ||\vec{v}|| + ||\vec{w}||$; we will prove this later, after we have introduced the notion of the (real-valued) *dot product* of two vectors and proved the important *Cauchy-Schwarz inequality* relating dot products to norms of vectors.

Historically, it took mathematicians a long time even to define $\mathbb{R}^n$ for $n > 3$; since there was no immediate geometric interpretation even of $\mathbb{R}^4$, it did not occur to anyone to define and study it. An interesting object that was introduced before $\mathbb{R}^n$ is the set H of *quaternions*, consisting by definition of all sums $a + bi + cj + dk$, where $a, b, c, d \in \mathbb{R}$. Such sums are added and subtracted in the "obvious" way; we decree that $(a_1 + a_2 i + a_3 j + a_4 k) \pm (b_1 + b_2 i + b_3 j + b_4 k) = (a_1 \pm b_1) + \ldots + (a_4 \pm b_4)k$. Matters become more interesting when we multiply quaternions. The idea is first that j, k are analogous to the complex number i, so that $i^2 = j^2 = k^2 = -1$. Next we decree that $ij = -ji = k, ki = -ik = j, jk = -kj = i$ (so that multiplication is in particular *noncommutative* for the quaternions). The best way to remember the multiplication rules is to visualize i, j, k as equally spaced around a circle; then the rule is that the product xy of any two of them equals the other one z if x, y are consecutive clockwise around the circle, while this product is $-z$ if x, y are consecutive counterclockwise. Now when we square the *pure* quaternion $a_i + bj + ck$ (so called since its real part is 0), we get $-a^2 - b^2 - c^2$, the negative of the square of the norm of (a, b, c) regarded as a vector in $\mathbb{R}^3$. Moreover, if we define the conjugate of the quaternion $a + bi + cj + dk$ as $a - bi - cj - dk$ (by analogy with conjugation of complex numbers), then the product $(a + bi + cj + dk)(a - bi - cj - dk)$ of any quaternion and its conjugate is $a^2 + b^2 + c^2 + d^2$, the square of the norm of the corresponding vector (a, b, c, d) in $\mathbb{R}^4$. We call this quantity the norm of the quaternion $z = a + bi + cj + dk$. Then it also turns out that if we write $N(z)$ for the norm of the quaternion z then we have $N(zw) = N(z)N(w)$:

norms satisfy a kind of product rule. As an exercise, see if you can work out all possible square roots of -1 in H. There are infinitely many of them!

Unfortunately this construction for $n = 4$ has no analogue for higher n (there is a partial analogue for $n = 8$, but nothing at all along these lines for any other value of n). In fact, the mathematician William Rowan Hamilton (who invented the quaternions and whose name accounts for the letter H used to denote them) tried for many years but failed to introduce a multiplication on $\mathbb{R}^3$ that would satisfy the above properties; it was only much later (and with great reluctance) that he realized one additional dimension was necessary, and that besides this a noncommutative multiplication was required. It is no coincidence, by the way, that the notations i, j, k used for the quaternions are also often used in physics to denote the standard unit coordinate vectors $(1, 0, 0), (0, 1, 0)$, and $(0, 0, 1)$ in $\mathbb{R}^3$. There is another kind of product for vectors in $\mathbb{R}^3$ (this time vector-valued) called the *cross product* whose definition will strongly remind you of multiplication in H. The dot and cross products were originally defined only for $\mathbb{R}^3$ in terms of quaternionic multiplication; it was realized only much later that the dot product can be defined for any $\mathbb{R}^n$. There is also a notion of "cross product" of $n - 1$ vectors in $\mathbb{R}^n$, which gives another vector in $\mathbb{R}^n$. We will indicate how to define it when we discuss determinants of $n \times n$ matrices next quarter.