Lecture 2-11

Convolutions arise in the solution to the *tautochrone problem*, which asks for the shape that a wire should have if a bead slides down it without friction in such a way that the time it takes to reach the bottom does not depend where on the wire it is released (p. 357 of the text). We begin with some remarks about this problem. At first it might seem impossible that different beads released simultaneously at different points along the wire could possibly reach the bottom at the same time, since beads released farther up have more distance to travel; but on further reflection we notice that beads released farther up are travelling at greater velocity when the reach the initial position of beads released lower down, so it is at least conceivable that the beads all take the same time to reach the bottom. Next, we might ask why a physicist might care about the answer to this problem. The answer comes from clock building. In the old days, the most accurate clocks were pendulum clocks; but all pendulum clocks lose energy over time, so that the arc through the bob at the end of the pendulum gradually gets shorter and shorter. In order for the pendulum not to lose time as this happens, we would like to know that the length of time it takes for the bob to descend from the highest point of its arc to the bottom does not depend on how high up the arc it goes. If the pendulum bob follows the arc of an inverted cycloid this will indeed be the case, as we show below.

Suppose that one end of the wire is at $(0,0)$ and the wire lies in the first quadrant. Let (a,b) be a point on the wire. For convenience we assume that the bead slides from right to left, starting at (a,b) and winding up at $(0,0)$. Using our usual coordinates x,y, and regarding the arclength s travelled by the bead as a function of y, we know that $f(y) = ds/dy$ is given by $\sqrt{1+(dx/dy)^2}$. The principle of conservation of energy shows that the time it takes for the bead to slide from (a,b) to $(0,0)$ is given by $T(b) = \frac{1}{\sqrt{2g}}\int_0^b \frac{f(y)}{\sqrt{b-y}}\,dy$, where g is the gravitational constant. Note that this last expression is exactly the convolution of $f(y)$ and $1/\sqrt{y}$. If $T(b)$ is a constant T_0, then taking Laplace transforms and using convolutions we get $F(s) = \sqrt{\frac{2g}{\pi}}\frac{T_0}{\sqrt{s}}$, whence $f(y) = \frac{\sqrt{2g}}{\pi}\frac{T_0}{\sqrt{y}}$. From the formula for $f(y)$ in terms of dx/dy, we get $\frac{dx}{dy} = \sqrt{\frac{2\alpha-y}{y}}$ where $\alpha = gT_0^2/\pi^2$. This last differential equation is separable, but rather than solving it directly it turns out to be easier to make the substitution $y = 2\alpha\sin^2(\theta/2)$. Then dx/dy works out to be $\cos(\theta/2)/\sin(\theta/2)$. Multiplying by $dy/d\theta$ we get $dx/d\theta = 2\alpha\cos^2(\theta/2) = \alpha(1+\cos\theta)$, while $y = 2\alpha\sin^2(\theta/2) = \alpha(1-\cos\theta)$. Integrating, we get $x = \alpha(\theta+\sin\theta), y = \alpha(1-\cos\theta)$. These are *not* quite the parametric equations of a cycloid, contrary to what is stated in the text, but if you start with the cycloid corresponding to a circle of radius a ($x = a(\theta-\sin\theta), y = a(1-\cos\theta)$), turn the graph upside-down (that is, reflect it about the line $y = a$), and then shift it by $a\pi$ units to the left or right, then you get the graph of the given parametrization; accordingly, we call this parametrized curve an inverted cycloid. Amazingly enough, the inverted cycloid also turns out to be the solution to the *brachistochrone problem*, which asks for the shape that the wire should have for the bead to slide from a fixed point on it to the bottom in the shortest possible time. We do not have the tools necessary to prove this (it requires techniques from the *calculus of variations*, which can be used to maximize or minimize expressions depending on an unknown function by in effect differentiating with respect to the function) but we did want to mention the remarkable coincidence of the solutions of these two problems.

We conclude this unit by mentioning that the Laplace transform is a special case of something called the *Fourier transform*, which attaches to every continuous function $f(x)$ the function $\hat{f}(t) = \int_{-\infty}^{\infty}(1/\sqrt{2\pi})f(x)e^{ixt}\,dx$ for $t \in \mathbb{R}$, defined whenever the integral converges. If we start with a function $f(x)$ defined only on $[0,\infty)$, extend it to all $x \in \mathbb{R}$ by decreeing that it be 0 for $x < 0$, and multiply by a suitable constant, then the Fourier transform of the resulting function, with the parameter t replaced by is, coincides with the Laplace transform of f. Recall also that we have defined the Fourier *coefficients* of a function $f(x)$ defined on $[-\pi,\pi]$ as the integral $a_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ or $b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$ for some n (replacing $1/\pi$ by $1/2\pi$ in the first integral if $n = 0$). Then under suitable hypotheses on $f(x)$ the Fourier series $\sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ converges to $f(x)$ for all $x \in [-\pi,\pi]$; note however that the Fourier series of $f(x)$ necessarily defines a periodic function with period 2π, so cannot converge to $f(x)$ for $|x| > \pi$ unless f also happens to be periodic. By contrast, the Fourier transform $\hat{f}(t)$ is not periodic in t, so that Fourier transforms can be used to yield important information about non-periodic functions that are not necessarily the sums of their Fourier series.

For the rest of the course we will focus on higher dimensions (than one) and return to Salas-Hille. We will see that *vector-valued functions* sending $t \in \mathbb{R}$ to the n-tuple $(f_1(t),\ldots,f_n(t))$ with the f_i differentiable can be handled fairly easily with the tools that we already have but real-valued functions $f(x_1,\ldots,x_n)$ of n real variables are considerably harder to understand; it will take substantially more work to decide when such functions are differentiable and what their derivatives are.