Lecture 2-10

We illustrate the use of Laplace transforms in solving equations involving Dirac delta functions. Consider the equation $2y'' + y' + 2y = \delta(t-5)$, together with the usual initial conditions $y(0) = y'(0) = 0$ (p. 346 in the text). Taking Laplace transforms, we get $(2s^2 + s + 2)Y(s) = e^{-5s}$, whence $Y(s) = \frac{e^{-5s}}{2s^2+s+2} = \frac{e^{-5s}}{2}\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}}$. Taking the inverse transform, we get $y(t)) = \frac{2}{\sqrt{15}}u_5(t)e^{-t-5)/4}\sin\frac{\sqrt{15}}{4}(t-5)$. Once again the solution is not actually twice differentiable at $t = 5$, but does have left and right second derivatives there, though this time we cannot compare these to the left and right second derivatives of $\delta(t-5)$ as $\delta(t-5)$ is not a function defined on numbers.

We have mentioned that there is no general formula for the Laplace transform $\mathcal{L}fg$ of the product of fg of two functions f, g in terms of the transforms of the factors; but curiously enough there *is* a formula for the *inverse* Laplace transform of a product. To derive it we need to define a new kind of "product" for functions. Given continuous functions f, g of a real variable, their *convolution* $h = f * g$ is defined by the formula $h(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau$; observe that (as with the Laplace transform) the integration takes place with respect to the variable τ and thereby defines a function of the other variable t. Note that convolution is easily seen to be commutative and distributive; it also turns out to be associative $((f * g) * h = f * (g * h))$; but there is no multiplicative identity for convolution; that is, there is no function g such that $f * g = g * f = f$ for all f. Also it is quite easy to show that the convolution $f * f$ of a function with itself need not be nonnegative. For example, the identity function $f(\tau) = \tau$, if convolved with itself, gives the function $\int_0^t \tau(t-\tau)\,d\tau = (-\frac{\tau^3}{3} + \frac{\tau^2}{2})|_0^t = t^3/6$.

Now we claim that *the Laplace transform of $f * g$ is the product $\mathcal{L}f\mathcal{L}g$ of the Laplace transforms of f and g.* To prove this, we look at the integral $\int_0^\infty \int_0^t f(t-\tau)g(\tau)e^{-st}\,d\tau\,dt$ defining $\mathcal{L}f * g$. This integral is a double integral taking place in the $t\tau$-plane, taking place over the region where $t \geq 0$ and $0 \leq \tau \leq t$. We can introduce a new coordinate $u = t - \tau$ and then describe this region by the inequalities $\tau \geq 0, u \geq 0$. Writing e^{-st} as $e^{-s\tau}e^{-su}$ we can then rewrite the integral as $\int_0^\infty \int_0^\infty e^{-s\tau}g(\tau)e^{-su}f(u)\,d\tau\,du$. Now the variables have been completely separated; evaluating each integral in turn with respect to its variable and treating the other variable as a constant, we get $\mathcal{L}f(s)\mathcal{L}g(s)$, as desired.

Using convolutions we get an explicit expression for the solution to a nonhomogeneous equation $ay'' + by' + cy = g(t)$ in terms of the forcing function $g(t)$ on the right side (p. 353 of the text). Imposing the initial conditions $y(0) = y_0, y'(0) = y_1$ and taking Laplace transforms, we get $(as^2 + bs + c)Y(s) - (as + b)y_0 - ay_1 = G(s)$; letting $\Phi(s) = \frac{(as+b)y_0+ay_1}{as^2+bs+c}, \Psi(s) = \frac{G(s)}{as^2+bs+c}$, we can write $Y(s) = \Phi(s) + \Psi(s)$, whence $y = \phi(t) + \psi(t)$, where ϕ, ψ are the respective inverse transforms of Φ, Ψ. Here $\phi(t)$ is our old friend the solution to the homogeneous initial-value problem $ay'' + by' + cy = 0, y(0) = y_0, y'(0) = y_1$, while $\psi(t)$ solves the inhomogeneous problem $y'' + ay' + by = g(t), y(0) = y'(0) = 0$. Writing $\Psi(s)$ as $H(s)G(s)$, where $H(s) = \frac{1}{as^2+bs+c}$ and $h(t)$ is its inverse Laplace transform, we can write $\psi(t) = h * g = \int_0^t h(t-\tau)g(\tau)\,d\tau$; this is equivalent to the formula we obtained before by variation of parameters, but is expressed in different terms, not explicitly involving any particular solutions to the homogeneous equation. We can think of $h(t)$ as the solution to the inhomogeneous problem $ay'' + by' + cy = \delta(t), y(0) = y'(0) = 0$; it is called the *impulse response*, while H is called the *transfer function*. In words, then, $\psi(t)$ is the convolution of the impulse response and the forcing function.