FINAL EXAM WITH SOLUTIONS

Math 135A

1. Let $f(x,y) = \frac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$ while $f(0,0) = 0$. Determine with proof whether or not f is differentiable at $(0,0)$. (First compute the partial derivatives of f at $(0,0)$ and then decide whether or not the definition of differentiability is satisfied with these values of the partial derivatives.)

We have $f_x(0,0) = \lim_{h\to 0}(f(h,0) - f(0,0))/h = \lim_{h\to 0} h/h = 1$; similarly $f_y(0,0) = 1$. To decide whether f is differentiable at $(0,0)$, we must determine whether or not $\lim \frac{h^3+k^3}{(h^2+k^2)(\sqrt{h^2+k^2})} = 0$; in fact, the limit does not exist, since this fraction approaches different limits depending on how (h,k) approaches $(0,0)$. Hence f is not differentiable at $(0,0)$.

2. Find a parametrization of the ellipse with equation $\frac{x^2}{4} + \frac{y^2}{9} = 1$ and integrate $x\,dy$ with respect to t over a suitable interval to find the area enclosed by this ellipse.

The standard parametrization is $x = 2\cos t, y = 3\sin t$. Computing the integral of $x\,dy$ as t runs from 0 to 2π (so that the point goes once counterclockwise around the ellipse), we get $\int_0^{2\pi} 6\cos^2 t\,dt = 6\pi$ for the desired area.

3. Find the set of values of x for which $\sum_{n=0}^{\infty}(x+4)^{-n}$ converges.

The series is geometric, so that it converges if and only if its common ratio $(x+4)^{-1}$ has absolute value less than 1. This occurs if and only if $x < -5$ or $x > -3$.

4. Given the linear differential equation $ay''+by'+cy = 0$ with positive constant coefficients a, b, c, show that the 0 solution is asymptotically stable (i.e. all solutions $y(t)$ approach 0 as $t \to \infty$).

The roots of the characteristic equation $ar^2 + br + c = 0$ are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, by the quadratic formula, so both of them have strictly negative real part under our hypothesis on a, b, c. Hence the 0 solution is asymptotically stable, by a result proved in class.

5. Use Laplace transforms to express the unique solution to $y^{(3)} - y = g(t)$ with $y(0) =$

$y'(0) = y''(0) = 0$ as a convolution integral. Here $g(t)$ is a continuous function.

Taking Laplace transforms, we get $(s^3 - 1)Y(s) = G(s)$, writing Y, G for the respective transforms of y, g. Writing $\frac{1}{s^3-1} = \frac{1/3}{s-1} - \frac{(1/3)s+(2/3)}{s^2+s+1} = \frac{1/3}{s-1} - \frac{1}{3}\frac{s+(1/2)}{(s+1/2)^2+(\sqrt{3}/2)^2} - \frac{1}{2}\frac{1}{(s+1/2)^2+(\sqrt{3}/2)^2}$, we get
$\int_0^t (\frac{1}{3}e^{(t-\tau)} - \frac{\sqrt{3}}{3}e^{-(1/2)(t-\tau)} \sin \frac{\sqrt{3}}{2}(t-\tau) - \frac{1}{3}e^{-(1/2)(t-\tau)} \cos \frac{\sqrt{3}}{2}(t-\tau))g(\tau)\, d\tau$ for the desired convolution integral.

6. Let L_1, L_2 be the lines through $(2, 5, -3)$ and $(4, 9, 5)$, respectively, with respective direction vectors $(3, 7, 2)$ and $(2, 6, -12)$. Determine whether or not L_1 and L_2 intersect, and if so at which point.

Solving the simultaneous equations $2 + 3s = 4 + 2t, 5 + 7s = 9 + 6t, -3 + 2s = 5 - 12t$, we get $s = 1, t = 1/2$, whence the lines intersect at the point $(5, 12, -1)$.

7. Let $\sum_{n=0}^{\infty} a_n x^n$ be a nonzero power series solution to the differential equation $y' - x^2 y = 0$. Say as much as you can about the coefficients a_n.

Solving the equation by separation of variables, we get $y = ce^{\frac{x^3}{3}}$, whence the desired power series solution is $\sum_{n=0}^{\infty} c\frac{x^{3n}}{3^n n!}$. Hence $a_n = 0$ if n is not a multiple of 3, while $a_n = \frac{c}{3^m m!}$ if $n = 3m$ is a multiple of 3.

8. Find the unit principal normal vector to the parametrized curve $(\cos t, \sin t, \cos 2t, \sin 2t)$.

The tangent vector is $(-\sin t, \cos t, -2\sin 2t, 2\cos 2t)$, whence the unit tangent vector is $1/\sqrt{5}$ times this vector. Differentiating again with respect to t and dividing by the length to make the resulting vector a unit vector, we get $\frac{1}{\sqrt{17}}(-\cos t, -\sin t, -4\cos 2t, -4\sin 2t)$.