Lecture 1-8

Continuing the discussion from last time, we now show how to measure how fast a curve is bending, independently of how it is parametrized. We do this by first recalling that the tangent line to a parametrized curve $(f(t), g(t))$ at the point where $t = a$ makes an angle $\arctan(y'/x')(a) = \arctan(g'(a)/f'(a))$ with the positive x-axis. Differentiating this with respect not to the original parameter t but rather with respect to the arclength s, we get $(|f'g'' - g'f''|/(f')^2)/(ds/dt) = |f'g'' - g'f''|/((f')^2 + (g')^2)^{3/2}$ evaluated at a, by the chain rule, applied twice; this measures how much the curve bends per unit of length when $t = a$ and is called the *curvature* κ of $(f(t), g(t))$ at $t = a$. Here we inserted the absolute value sign since curvature is usually taken to be nonnegative. In the case of a graph $(t, f(t))$ of a function $y = f(x)$, the curvature at $t = a$ reduces to $|f''(a)|/(1 + f'(a)^2)^{3/2}$. Notice that we were able to derive this formula without first reparameterizing the curve by arclength, and in fact without using any explicit expression for the arclength at all.

In particular, given the circle $(a\cos t, a\sin t)$ of radius a centered at the origin, the formula for its curvature at any point quickly reduces to $a^2/a^3 = 1/a$: circles bend at a constant rate equal to the reciprocal of their radii at any point. That circles have a constant curvature should be intuitively clear; that this curvature should increase as the radius decreases and conversely should be equally clear, since we are measuring the rate of change of direction with respect to arclength and small circles have small circumferences. In physical terms, if you imagine yourself in a car proceeding at a constant speed over a circular racetrack, the acceleration you feel in your body is greater for a small track than for a large one for any given speed. We will quantify this difference and relate curvature to velocity and acceleration later, when we will also extend the definition of curvature (and define another quantity called *torsion*) for curves in $\mathbb{R}^3$ rather than $\mathbb{R}^2$. For now, we just note that in fact the *only* (plane) curves with constant nonzero curvature are arcs of circles, while the only curves with curvature zero are straight line segments. You might enjoy verifying this last fact on your own, by setting $f''(x)/(1 + f'(x)^2)^{3/2}$ equal to a positive constant k and solving this equation for $f(x)$. No techniques of differential equations beyond antidifferentiation are required to do this, but a number of substitutions are needed. The upshot is that the graph of $f(x)$ has to be an arc of a circle of radius $1/k$, but of course the center of the circle can be any point in the xy-plane.

We now return to the special case of parametrized curves considered last quarter, namely polar graphs. Given the polar equation $r = f(\theta)$, we have seen that the corresponding parametrized curve is $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta)$. Computing the slope y'/x' of the tangent line, we get $\frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta)}$. In particular, for the circle $(a\cos\theta, a\sin\theta)$ of radius a centered at the origin, we get $y'/x' = -a\cos\theta/a\sin\theta = -\cot\theta = -x/y$ as usual. Computing the speed of a polar parametrized curve $(x, y) = (f(\theta)\cos\theta, f(\theta)\sin\theta$ we get $\sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{f(\theta)^2 + f'(\theta)^2}$, whence the arclength of the segment of this curve for $\theta \in [a, b]$ is given by $\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$. Note that this formula, by contrast to the earlier formula for the area enclosed by a polar graph, applies to any interval $[a, b]$; it does not matter how long the interval is. Thus the arclength of the cardioid $r = a(1 - \cos\theta)$ with $\theta \in [0, 2\pi]$, for example, is given by what turns out to be a familiar integral, namely $\int_0^{2\pi} \sqrt{(a^2((1 - \cos\theta)^2 + \sin^2\theta)} = \int_0^{2\pi} 2a\sin\frac{1}{2}\theta = 8a$. We saw this same integral when computing the arclength of the cycloid over the same interval for θ. Note also that we get the very simple formula $\sqrt{2}(e^b - e^a)$ for the arclength of the exponential spiral $r = e^\theta$ between the rays $\theta = a$ and $\theta = b$. The corresponding integral for the arclength of the ordinary spiral $r = \theta$ is more complicated, requiring a trigonometric substitution and the tricky formula for the antiderivative of $\sec\theta$.

The formula $\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$ for the arclength of a polar graph segment, when combined with the formula $\int_a^b (1/2)f(\theta)^2\, d\theta$ for the area of the corresponding polar region, suggests that of all regions corresponding to closed polar graphs (i.e., closed as curves) with a fixed perimeter, the circle has the largest area, since if $\int_a^b \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$ is fixed, then one maximizes the contribution made by $f(\theta)$ to this integral (and thus the contribution of $f(\theta)$ to the area integral) by setting $f'(\theta)$ equal to 0; that is, by taking $f(\theta)$ to be constant. Of course this intuition *proves* nothing, and indeed the polar graph of the nonconstant function $r = \cos\theta$, which is also a circle, encloses the largest possible area of any closed curve with its perimeter as well. Nevertheless, it is interesting to observe that the correct solution to this very difficult max-min problem (belonging to something called the *calculus of variations*) is suggested by a pair of rather simple formulas.

The formula $\int_a^b x\, dy$ given on Monday for the area of a region enclosed by a curve $(x(t), y(t))$ traced once counterclockwise as t runs over the interval $[a, b]$ can be used to compute the area of the ellipse with equation $(x^2/a^2) + (y^2/b^2) = 1$, which turns out to be πab. By contrast, the arclength of (the boundary of) this ellipse is given by an integral which cannot be evaluated. It is important enough however to deserve a special name (and indeed an entire branch of calculus is devoted to it); not surprisingly it is called an elliptic integral.