Lecture 1-7

Continuing with parametrized curves, let $C = \{(f(t), g(t) : t \in [a, b]\}$ be such a curve with f and g have continuous derivatives on $[a, b]$. In order to define the arclength of C, it is reasonable to start with a partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$, say one with equal lengths for simplicity, look at the sum $\sum_{i=1}^{n} \sqrt{(f(t_i) - f(t_{i-1})^2 + (g(t_i) - g(t_{i-1})^2}$ of the lengths of the line segments joining $(f(t_{i-1}, g(t_{i-1}))$ to $(f(t_i), g(t_i))$ for $1 \le i \le n$, and take the limit as this sum as $n \to \infty$. Recalling that each subinterval of P has length $(b-a)/n$ and applying the Mean-Value Theorem, we can rewrite the sum as $\frac{b-a}{n} \sum \sqrt{f'(u_i)^2 + g'(v_i)^2}$, for some u_i, v_i lying between t_{i-1} and t_i. While the resulting sum is not quite a Riemann sum for $I = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$ (since u_i, v_i can differ), it comes close enough that we can identify the limit of this sum as the integral for I; we therefore define the arclength of C to be the integral $\int_a^b \sqrt{f'(t)^2 + g'(t)^2}\, dt$. The physical interpretation of the integrand $\sqrt{f'(t)^2 + g'(t)^2}$ is the *speed* of a particle moving so that its position at time t is $(f(t), g(t))$; then the arclength in question is just the distance travelled by the particle as t runs over $[a, b]$. In particular, the graph segment of a function $y = f(x)$ between $x = a$ and $x = b$ has arclength $\int_a^b \sqrt{1 + f'(x)^2}\, dx$, assuming $f(x)$ has a continuous derivative on $[a, b]$. The ordered pair $(f'(t), g'(t))$ is called the *velocity* or *tangent vector* to the curve; it measures how fast both the x- and y-coordinates of a point travelling along it are changing at any time.

Of course the presence of the square root sign in the integrand makes the integral difficult or impossible to evaluate in most examples. Fortunately, as we will see below, the expression for the derivative of the arclength is sufficient for many applications; one does not need an expression for the arclength itself. We will look at two families of curves, each indexed by a single parameter, which are such that only one value of the parameter allows the arclength integral to be computed (apart from trivial cases). In the first case, we let c be a positive constant and consider the family of graphs $(t, d\cosh ct)$ of graphs of functions $y = d\cosh ct$, where d is another (nonzero) constant. The arclength integrand is then $\sqrt{1 + c^2 d^2 \sinh^2 ct}$: only if $d = 1/c$ can we evaluate the integral. In that case the integrand simplifies to $\sqrt{1 + \cosh^2 ct} = \sinh ct$ and the arclength between $t = 1$ and $t = b$ is just $\frac{\cosh cb - \cos ca}{c}$. Amazingly enough, this one computable case turns out to be the most important one in physics: the shape of a homogeneous, flexible, inelastic rope hanging between two fixed points is given by a graph of this form for some c (called a *catenary*; see Exercise 10.7.54).

In the second case we consider the path traced by a point on a wheel of radius a that rolls without slipping down the positive x-axis, starting on the y-axis. We can work out the equation of the point's motion by breaking it down into two simpler motions. The center of the wheel traces the ray corresponding to the graph of $y = a$ for $x \ge 0$, so that its position at time t is (at, a). Meanwhile, the point traces a circle of radius b clockwise around the center, starting at distance $b \le a$ directly below it; its position at time t relative to the center is $(-b\sin t, -b\cos t)$. Putting the two motions together (in physical terms, superposing them), we get the formula $(at - b\sin t, a - b\cos t)$ for the position of the point at time t. Here the integrand for the arclength is $\sqrt{a^2 + b^2 - 2ab\cos t}$. Apart from the trivial case $b = 0$ the only value of b leading to a computable integral is a; in this case the integrand $\sqrt{2a^2(a - \cos t)}$ simplifies to $2a \sin(1/2)t$, so that the distance travelled by the point between $t = 0$ and $t = t_0$ is $4a(-\cos(t_0/2) + 1)$, or $8a$ if the point goes all the way around the wheel once. The curve is called a *cycloid* in this case and a *trochoid* in general (if $b < a$). Once again, we are most fortunate that cycloids turn out to be much more important than trochoids. We will return to cycloids and closely related curves called inverted cycloids later; the latter curves provide a solution to a famous 17th century problem in physics called the *brachistochrone* problem, which asks for the shape that a frictionless wire joining two fixed points should have for a bead sliding along it and acted on only by gravity to get to the bottom point from the top one in the shortest time.

Given a curve with parametrization $(f(t), g(t))$ for $t \in [a, b]$, we might be interested in other curves traversing the same set of points in the same order. Such curves are given by the parametrization $f(h(s), g(h(s))$ for $s \in [c, d]$, where h is a differentiable strictly increasing function taking the interval $[c, d]$ to $[a, b]$ (so that $h(c) = a, h(d) = b$). Computing $(f(h(s))', g(h(s))')$ by the chain rule, we get $h'(s)(f(h(s), g(h(s))$; thus *both coordinates of the tangent vector at any point of the curve are replaced by the same positive constant multiple of themselves* and in particular the slope of the tangent line to the curve at any point remains unchanged. If instead h is strictly decreasing, then the same calculation applies, except that the multiplying constant $h'(s)$ is negative rather than positive. Thus the slope of the tangent line to a curve depends on the curve only as a set of points and a direction of motion; the only difference between two parametrizations of the same set of points in the same order comes from the speed of the moving particle at a given time. For example, the curve given by $(\cos t, \sin t)$ traces the unit circle once counterclockwise at constant speed equal to 1 as t runs over $[0, 2\pi]$; the curve given by $(\cos t^2, \sin t^2)$ traces the same circle once counterclockwise, but this time at steadily increasing speed, as t runs over $[0, \sqrt{2\pi}]$. On the other hand, the pair $(\cos t, -\sin t)$ traces the same circle at unit speed, but this time clockwise.

In particular, we can give an intrinsic parametrization of any curve given the set of points on it and a direction of motion. To do this we fix one point arbitrarily as the initial one and decree that a particle start at this point and move in such way that at time t it has moved distance exactly t along the curve. A curve parametrized in this way is said to be *parametrized by arclength*; we usually label the parameter by s rather than t in this case. Thus a curve is parametrized by arclength if and only if its speed is constantly equal to 1. We can study the geometry of the set of points corresponding to this curve by differentiating the angle between its tangent line and the x-axis with respect to arclength. We do this next time.