Lecture 1-6

We begin the course by looking at parametrized curves more systematically than we did last quarter, following §§10.5-7 of the text. Given any two differentiable functions $f(t), g(t)$ on a closed interval $[a, b]$, we call the set of points $\{(f(t), g(t)) : t \in [a, b]\}$ a parametrized curve; note that the functions $f(t), g(t)$ and not just this set are included in the definition and the curve is typically labelled by the ordered pair $(f(t), g(t))$. We say that two parametrized curves $(f(t), g(t)), (h(t), k(t))$ *intersect* at a point (a, b) if there are t_1, t_2 such that $(f(t_1), g(t_1)) = (h(t_2), k(t_2)) = (a, b)$; we say that the curves *collide* at (a, b) if there a single t such that $f(t), g(t)) = (h(t), k(t)) = (a, b)$. Thus, following Example 6 on p. 500 of the text, we compute that the curves $(\frac{16}{3} - \frac{8}{3}t, 4t - 5)$ and $(2\sin\frac{\pi t}{2}, -3\cos\frac{\pi t}{2})$ intersect at $(0, 3)$ and $(2, 0)$, as one sees by sketching both curves (the first one is a line, the second one an ellipse centered at $(0, 0)$). These curves do not collide at $(2, 0)$, however, since the first one passes through that point at $t = 5/4$ while the second does not pass through that point at that time. On the other hand, both curves pass through their other intersection point $(0, 3)$ at the same time $t = 2$, so they collide there.

There are standard parametrizations of the conic sections. Given an ellipse in standard position, with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, it can be parametrized via $(x, y) = (a\cos t, b\sin t)$ (but note that t here is *not* the same as θ in polar coordinates). Similarly, the hyperbola with equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has the parametrization $(x, y) = (\pm a\cosh t, b\sinh t)$ (each sign corresponding to a different branch); this is why the hyperbolic trig functions are so named. The parabola with equation $y^2 = 4cx$ admits the obvious parametrization $(x, y) = (\frac{t^2}{4c}, t)$.

Given a closed curve $C = (f(t), g(t))$ as t runs over an interval $[a, b]$, so that we have $(f(a), g(a)) = (f(b), g(b))$, it is natural to wonder whether the area enclosed by the curve can be computed by an integral; note that a polar graph defined by the equation $r = f(\theta)$ and the interval $\theta \in [a, b]$ becomes a closed curve if line segments from $(0, 0)$ to $(f(a)\cos a, f(a)\sin a)$ and to $(f(b)\cos b, f(b)\sin b)$ are added to it. One might expect the area enclosed by C to be given by $\int_a^b y\,dx = \int_a^b g(t)f'(t)\,dt$, by analogy with the area under the graph of a function $\phi(x)$ and between two fixed limits for x. If we try this out on a familiar example, namely the unit circle $(x, y) = (\cos t, \sin t)$ with $t \in [0, 2\pi]$, then we get the curious answer $\int_0^{2\pi} -\sin^2 t\,dt = -\pi$ for the area of the circle. Thus our intuition is not quite correct here; but the very fact that we are off by exactly a sign suggests that this intuition is not useless; we must be onto something. Looking more carefully at how our parametrization traces the circle, we notice that that t runs from 0 to 2π, we trace the upper semicircle from right to left (in the direction of *decreasing* x, while the lower semicircle is traced in the direction of increasing x. Thus it is not surprising after all that we got the negative of the true area. Had we integrated $x\,dy$ over the same limits, then the right semicircle would be traced in the direction of increasing y (over the intervals $[0, \pi/2]$ and $[3\pi/2, 2\pi]$, while the left semicircle is traced in the direction of decreasing y. Thus we would expect to get the true area in this case, and indeed $\int_0^{2\pi} \cos^2 t\,dt = \pi$. The rule in general is that *if the parametrization $(f(t), g(t))$ for $t \in [a, b]$ traces the boundary of a closed curve counterclockwise, then the area enclosed by the curve is $\int x\,dy = \int_a^b f(t)g'(t)\,dt$; but if this parametrization traces the boundary curve clockwise, then the area is given by*

$\int_y dx = \int_a^b g(t)f'(t)\,dt$. This rule is a special case of a general theorem called Green's Theorem, which strictly speaking does not lie in the syllabus of the 13x sequence (it is covered in Math 324), but which is treated in more detail in Chapter 18 of the text.

We have already seen the formula for the slope of the tangent line to a parametrized curve $(f(t), g(t)$ corresponding to $t = t_0$; it is $g'(t_0)/f'(t_0)$, provided that $f'(t_0) \neq 0$. Note that it is important to specify the value t_0 of the parameter t and not just the point $(f(t_0), g(t_0))$, since a single parametrized curve can pass through the same point at several times, with a different slope of the tangent line each time. For example, the curve $(t(t-2\pi), \sin t)$ passes through the origin $(0,0)$ twice, once at time $t = 0$ and again at time $t = 2\pi$. The slope of the tangent line when $t = 0$ is $-1/2\pi$, while the slope of this line when $t = 2\pi$ is $1/2\pi$. If $f'(t_0) = 0$ but $g'(t_0) \neq 0$, then we (not surprisingly) decree that the tangent line for $t = t_0$ is vertical. What if $f'(t_0) = g'(t_0) = 0$? We have already seen this situation in the case of the folium of Descartes and the lemniscate. Now, for the first time in the course, we find ourselves having to evaluate limits of the form $\lim_{t\to a} f(t)/g(t)$ in situations where $\lim_{t\to a} f(t) = \lim_{t\to a} g(t) = 0$ by a general rule, not relying on the definition and elementary properties of limits. The answer, as you all know, is given by L'Hopital's Rule: $\lim_{x\to a} f(x)/g(x) = \lim_{x\to a} f'(x)/g'(x)$, provided this last limit exists. This follows since if we define $f(a) = g(a) = 0$, then the ratio $f(x)/g(x) = (f(x) - f(a)/(g(x) - g(a)) = f'(c)/g'(c)$ for some c between x and a, by Cauchy's mean-value theorem (proved in a HW problem last term), whence $f(x)/g(x)$ has the same limit as $f'(x)/g'(x)$ does as $x \to a$, provided the latter limit exists. L'Hopital's Rule also holds (as you no doubt know) for limits as $x \to \infty$ (replace $f(x), g(x)$ by $f(1/x), g(1/x)$, and let x go to 0), or for limits $f(x)/g(x)$ if $f(x), g(x)$ both go to ∞ as $x \to a$ (replace $f(x)/g(x)$ by $(1/g(x)/(1/f(x)))$. In particular, by repeated application of this rule, we see that $p(x)/e^x \to 0$ as $x \to \infty$ for any polynomial $p(x)$, a fact used in an example last quarter.