Lecture 1-29

We now review the material covered so far for the midterm on Friday. This will concentrate primarily on series, particularly power series, together with the existence-uniqueness theorem for first-order linear and nonlinear equations and exact equations.

By definition any series $\sum_{n=1}^{\infty} a_n$ is a certain kind of sequence, namely the sequence of its partial sums $s_n = \sum_{n=1}^{m} a_n$; thus the series converges if and only if s_n has a finite limit as $n \to \infty$. The basic examples where this limit can be computed explicitly are the geometric series $\sum_{n=0}^{\infty} ar^n$, which converges (for $a \neq 0$ exactly when $|r| < 1$, to $a/(1-r)$, and the telescoping series $\sum_{n=1}^{\infty}(b_n - b_{n+1})$ for some sequence (b_n), which converges exactly when the sequence (b_n) does, to $b_1 - (\lim_{n\to\infty} b_n)$. We work out the convergence behavior of other series by comparing them to one of these series. Thus if $\sum a_n, \sum b_n$ both have nonnegative terms and $a_n \leq b_n$ for sufficiently large n, then $\sum a_n$ converges whenever $\sum b_n$ does and $\sum b_n$ diverges whenever $\sum a_n$ does. In particular, we get another proof that the series $S = \sum_{n=1}^{\infty}(1/n^2)$ converges: comparing the series $\sum_{n=2}^{\infty}(1/n^2)$ consisting of all terms but the first of S to the telscoping series $\sum_{n=1}^{\infty} 1/n(n+1)$, which converges to 1, we deduce that S converges to a number less than $1+1 = 2$ (in fact, to $\pi^2/6$). We also have the integral test, which says that given a series $S = \sum a_n$ with nonnegative terms such that $a_n = f(n)$ for some continuous nonnegative function $f(x)$ decreasing to 0 as $x \to \infty$, the seres S converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ does. Using either this test or the Cauchy condensation test, we get that the p-series $\sum_{n=1}^{\infty}(1/n^p)$ converges if and only if $p > 1$.

The ratio and root tests provide convenient methods for deciding whether series converge absolutely without having to compare them to other series; recall that a series converges whenever it converges absolutely. Given a series $\sum a_n$, the ratio test asserts that it converges absolutely whenever $\lim |a_{n+1}/a_n| < 1$ and diverges whenever $\lim |a_{n+1}/a_n| > 1$; similarly the root test asserts that $\sum a_n$ converges absolutely whenever $\lim |a_n|^{1/n} < 1$ and diverges whenever $\lim |a_n|^{1/n} > 1$. The case where the limit equals 1 is an indeterminate one for both tests; they fail in this case to give definitive information. One of the main applications of the ratio test is to power series $\sum a_n x^n$ and Taylor series $\sum a_n(x-a)^n$. Recall first that any power or Taylor series has a *radius of convergence* R, so that if converges absolutely if $|x|$ or $|x-a|$ is less than R and diverges if $|x|$ or $|x-a|$ is larger than R. (The cases where $|x|$ or $|x-a|$ equals R are deliberately left ambiguous; we allow either convergence or divergence in these cases.) Given a power or Taylor series $\sum a_n x^n$ or $\sum a_n(x-a)^n$ such that $\lim |a_{n+1}/a_n| = L$, its radius of convergence R equals $1/L$, where we agree for this purpose that $R = 0$ if $L = \infty$ and conversely.

Any power or Taylor series with a positive radius of convergence defines an infinitely differentiable function within the radius of convergence, whose derivative has a series obtained from the original one by differentiating each term; thus if $f(x) = \sum a_n(x-a)^n$ for $|x-a| < R$ and $R > 0$, then $f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$. Also any Taylor series can be integrated term by term: under the same hypothesis on f, we have $\int_a^x f(t)\,dt = \sum_{n=0}^{\infty}(a_n/n+1)(x-a)^{n+1}$. For any infinitely differentiable function $f(x)$ its Taylor series at $x = a$ is defined to be $\sum_{n=0}^{\infty} f^{(n)}(a)/n!(x-a)^n$ (whether or not this series converges to $f(x)$). If $f(x)$ is the sum of its Taylor series in some interval $(a-R, a+R)$ with $R > 0$, then we call $f(x)$ analytic at $x = a$. Some very wellknown examples of analytic functions are $e^x = \sum_{n=0}^{\infty} x^n/n!, e^{-x} = \sum_{n=0}^{\infty}(-1)^n x^n/n!$, and $\sin x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!, \cos x = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$; in all cases the radius of convergence is infinite. Examples of power series with a finite radius of convergence include $1/(1-x)\sum_{n=0}^{\infty} x^n$ and $-\ln(1-x) = \sum_{n=0}^{\infty} x^{n+1}/(n+1)$, both with radius of convergence 1, and $\arctan x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$, also with radius of convergence 1. Whenever a power or Taylor series with a finite radius of convergence happens to converge at one or both endpoints of the interval of convergence, it always converges to the "right" value, that is, the limit of its values as x approaches the endpoint. Thus, since the series of $\ln(1+x)$, namely $\sum_{n=1}^{\infty}(-1)^{n+1}x^n/n$, converges at $x = 1$ (by the alternating series test), its sum there must be the limit of $\ln(1+x)$ as $x \to 1^-$, namely $\ln 2$. Similarly, we must have $\pm\pi/4 = \pm\sum_{n=0}^{\infty}(-1)^n/(2n+1)$.

Turning now to differential equations, the first (and hardest) new result we have seen this quarter is the *existence-uniqueness theorem*: given the initial-value problem $y' = f(t,y), y(t_0) = y_0$, if both $f(t,y)$ and its y-partial $f_y(t,y)$ exist and are continuous on some rectangle R having (t_0, y_0) in its interior, then there is a unique solution to this problem, whose graph exists up to the boundary of R. Here the y-partial f_y is obtained from f by differentiating (as usual) with respect to y, treating t as a constant. In particular, the solution must be defined at least on some open interval $(t_0 - \epsilon, t_0 + \epsilon)$ containing t_0. In the case of a first-order linear equation $p(t)y' + q(t)y = r(t)$, one can say a little more: given any initial condition $y(t_0) = y_0$, the unique solution must be defined on any interval on which p, q, r are all defined and continuous and the coefficient $p(t)$ of y' is not 0.

We also learned that an *exact equation*, that is, one of the form $p(x,y)\,dx + q(x,y)\,dy = 0$, where there is a function $f(x,y)$ such that $f_x = p, f_y = q$, admits the general solution $f(x,y) = c$ where c is an arbitrary constant. In turn, to determine whether there is an f with $f_x = p, f_y = q$, it suffices in most cases to compute whether $p_y = q_x$.