Lecture 1-28

We will now be spending some time on differential equations, supplementing the material we learned last quarter from Salas-Hille with new material from the new textbook, Boyce-DiPrima. This will take up much of February; we will return later to Chapters 13 and 14 of Salas-Hille.

We saw last time that the initial-value problem $y' = f(t, y), y(t_0) = y_0$ always has a unique solution whenever $f(t, y)$ and its y-partial $f_y(t, y)$ exist and are continuous in a rectangle $[a, b] \times [c, d]$ containing the point (t_0, y_0) in its interior $(a, b) \times (c, d)$; the graph of this solution exists up to the boundary of $[a, b]\times[c, d]$. We now observe that the hypotheses on f are necessary to guarantee the uniqueness of the solution; the equation $y' = 3y^{2/3}$ coupled with the condition $y(0) = 0$ admits two solutions, namely $y(t) = t^3$ and $y(t) = 0$. In fact, this problem admits infinitely many solutions: for any $c \geq 0$ the function $f_c(t)$ defined to be 0 if $t \leq c$ and $(t - c)^3$ if $t \geq c$ is a solution. The problem here is that the y-partial of $3y^{2/3}$ blows up at $y = 0$, so there is no constant M with $|3y^{2/3}| = 3y^{2/3} \leq M|y|$ for all y. The other hypothesis (that $f(x, y)$ is continuous) is clearly also necessary as well even to guarantee the existence of one solution, since for example most discontinuous functions $f(t)$ even of just one variable t are not derivatives, so that the equation $y' f(x)$ as no solution. Another theorem (which we will not prove) asserts that the above initial-value problem always has at least one solution whenever $f(t, y)$ is continuous on $[a, b] \times [c, d]$.

Now we want to generalize the separable first-order equations that we saw last quarter to a larger class of equations which can be solved in much the same way. Recall first that given any function $f(x, y)$ of two variables such that f is differentiable with respect to x for each fixed y and differentiable with respect to y for each fixed x, we write $f_x = \partial f/\partial x$ and $f_y \partial f/\partial y$ for the derivatives of f with respect to x and y. These are called the *partial derivatives* (or just *partials*) of f and will be used later to determine which functions $f(x, y)$ are differentiable as functions of two variables and what their derivatives are. For now we just mention that the chain rule we saw last quarter has an analogue for functions of several variables: given a function $F(x, y)$ whose x- and y-partials exist, suppose that y is itself a differentiable function of x, so that the composite function $F(x, y(x))$ is a a function of x alone. Then we have the formula $dF/dx = F_x + F_y(dy/dx)$ for the derivative of F with respect to x. In particular, if the function $y(x)$ is such that $F(x, y(x))$ is constant, then we must have $F_x + F_y(dy/dx) = 0$, so that $dy/dx = -F_x/F_y$ at any point $(x, y(x))$ such that $F_y(x, y(x)) \neq 0$. Now suppose we are given the differential equation $y' = g(x, y)/h(x, y)$. If there is a function $F(x, y)$ such that $F_x = -g(x, y)$ and $F_y = h(x, y)$, then the implicit equation $F(x, y) = c$ for some constant c, assumed to define y uniquely as a function of x (subject to the further requirements that $F(a, b) = c$ for specified numbers a, b and that $y(x)$ be close to b if x is close to a), will then be such that $y' = g(x, y)/h(x, y)$, as desired; moreover, any solution $y(x)$ to this equation must have $dF(x, y(x)/dx = 0$, so that $F(x, y(x)$ must be a constant. The upshot is that we have found the general solution to the equation $y' = g(x, y)/h(x, y)$, since for any specified point (x_0, y_0) the implicit equation $F(x, y) = F(x_0, y_0)$ defines the unique solution to our equation satisfying the initial condition $y(x_0) = y_0$. By the way, the original equation $y' = -g(x, y)/h(x, y)$ is often rewritten in this situation as $g(x, y)\, dx + h(x, y)\, dy = 0$.

We are thus led to the question of deciding for a given pair of functions $g(x, y), h(x, y)$

whether there is a third function $F(x,y)$ such that $F_x = g, F_y = h$. Any differential equation $y' = -g(x,y)/h(x,y)$ for which there is a function F with this property is called *exact*. Now it turns out that the *second-order* partials F_{xy}, F_{yx} of any differentiable function $F(x,y)$ of two variables, obtained respectively by partially differentiating F_x and F_y with respect to y and x, are equal whenever both are continuous. Hence a fundamental necessary condition that there be an F with $F_x = g(x,y)$ and $F_y = h(x,y)$ is that $g_y(x,y) = h_x(x,y)$. In most cases this necessary condition is also sufficient. For example, given $g(x,y) = y, h(x,y) = x + y^2$, we find that $g_y = h_x = 1$, so we expect that there is an F with $F_x = g, F_y = h$. To find F, start by integrating g with respect to x to get $F_1 = xy$. Now the y-partial $(F_1)_y = x$, so we need to add a function $f(y)$ of y alone to F_1 (so that it does not affect the x-partial of F_1) to obtain the F we are looking for. A suitable choice is $f(y) = y^3/3$, so that $F(x,y) = xy + y^3/3$ has the desired properties. Thus the exact equation $y\,dx + (x + y^2)\,dy = 0$ has the general implicit solution $F(x,y) = xy + y^3/3 = c$ for some constant c.

In particular, any separable equation $f(x)\,dx + g(y)\,dy = 0$ is automatically exact, for given antiderivatives $F(x), G(y)$, of f, g, respectively, we have $H_x = f(x), H_y = g(y)$, where $H(x,y) = F(x) + G(y)$. Thus the discussion here completely recovers what we said about separable equations last quarter.