Lecture 1-24

We wrap up power series with a practical discussion of how they can be used to approximate the values of functions at points to a specified accuracy; this is what your calculator has to do when you punch the appropriate buttons on it, with the accuracy required being governed by the size of the display. Given any $(n+1)$-times differentiable function and a point a, we have the nth *Taylor polynomial of f and a*, defined to be the nth partial sum $T_n f = \sum_{i=0}^{n}(f^{(i)}(a)/i!)(x-a)^i$ of the Taylor series of f at a. Without necessarily having to assume that f is the sum of its Taylor series, we can ask how closely $T_n f(x)$ approximates $f(x)$ for x near a. The most convenient answer for most purposes is given by *Taylor's formula with the Lagrange remainder*, which asserts that $f(x) = T_n f(x)+R_{n+1}(x)$, where the nth *remainder term* $R_{n+1}(x) = (f^{(n+1)}(c)/(n+1)!)(x-a)^{n+1}$ for some c between a and x; note that the special case $n = 0$ is just the Mean-Value Theorem. If we have a bound for $f^{(n+1)}(x)$ for x in a suitable interval around a, then we can use this remainder to fund an upper bound for the error $|f(x)-T_n f(x)|$ in this interval. For example, if $f(x) = e^x$ and $a = 0$, then of course $f^{(n)}(x) = e^x$, an increasing function, for all nonnegative integers n, whence we can take e^{a+d} to be an upper bound for $f^{(n)}(x)$ on any interval $[a-d, a+d]$. Thus we can estimate say e^1 as the sum $\sum_{n=0}^{6} 1/n!$ of the first seven terms of Taylor expansion at $a = 0$ with an error at most $e/7! < 3/7! = 1/1680 < 0.0006$, that is, with an error of at most 6 in the fourth decimal place. (The actual error is less than 3 in the fourth decimal place.) More generally, the same argument shows that $T_n e^x$ approximates any e^X with an error of at most $(e^X/(n+1)!)X^{n+1}$ (for $X > 0$); since this last quantity goes to 0 as n goes to infinity for any fixed X, we get a new proof that e^x is the sum $\sum_{n=0}^{\infty} x^n/n!$ of its Taylor series for any $x > 0$, which easily extends to show that the same is true for any $x \in \mathbb{R}$. This is actually the approach taken by this text and most others to show that the exponential function e^x is analytic and that its power series has infinite radius of convergence (in §12.6); but I have preferred to treat Taylor series before Taylor polynomials as most students find the former more elegant and intuitively appealing.

A similar but more powerful argument shows that $\sin x$ and $\cos x$ are also both analytic at $x = 0$ (or any other point); it is more powerful since we have the easier and uniform upper bound of 1 for any $|f^{(n)}(x)$ if $f(x) = \sin x$ or $f(x) = \cos x$. We also see (for example) that $\sin 0.5$ is approximated by the sum $0.5 - (0.5)^3/6 = 23/48$ of just the first two terms of its Taylor series with an error of less than 0.001; see Example 7 in the text on p. 610. Note also that the sum of any alternating series $\sum_{n=0}^{\infty}(-1)^n a_n$ with $a_n \geq a_{n+1} \geq \ldots \to 0$ as $n \to \infty$ is overestimated by any odd partial sum $\sum_{n=0}^{2m}(-1)^n a_n$ and underestimated by any even partial sum $\sum_{n=0}^{2m+1}(-1)^n a_n$, with error bounded by the absolute value of the next term in both cases. This last bound is sometimes better than the one coming from the Lagrange remainder.

We conclude our treatment of power series with a brief look at another way to represent functions as infinite sums, namely as Fourier series. We have defined these already, at least in a special case; in general, given a continuous function $f(x)$ on the interval $[-\pi, \pi]$, set $a_n = (1/\pi)\int_{\pi}^{\pi} f(x)\cos nx\,dx$ for $n > 0$ and $a_0 = (1/2\pi)\int_{-\pi}^{\pi} f(x)\,dx$; for $n > 0$ we set $b_n = (1/\pi)\int_{-\pi}^{\pi} f(x)\sin nx\,dx$. The series $\sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ is called the

Fourier series of $f(x)$; note that it is defined for any continuous function f, so that one can potentially represent much larger class of functions by Fourier series than by power series. This potential is in fact realized; although not every continuous function is the sum of its Fourier series at every $x \in [-\pi, \pi]$ there are rather weak conditions that guarantee this; for example, every differentiable function on $(-\pi, \pi)$ is the sum of its Fourier series at any point in this open interval. There are some caveats, however; perhaps the most serious one is that Fourier series are set up in such a way that they can only represent *periodic* functions with period 2π. Thus the Fourier series of a differentiable function $f(x)$ on $[\pi, \pi]$ converges to $f(\pm\pi)$ at the endpoints $\pm\pi$ only if $f(\pi) = f(-\pi)$ (in general, the Fourier series of any such $f(x)$ converges at $\pm\pi$ to the average $(1/2)(f(-\pi) + f(\pi))$. Something similar is going on with the example $\sum_{n=1}^{\infty} \sin(nx)/n$ we mentioned earlier; this sum converges to $(\pi - x)/2$ for $x \in (0, \pi]$ and to $(x + \pi)/2$ for $x \in [-\pi, 0)$, but *not* at $x = 0$; what happens there is that the series converges to the average of the right- and left-hand limits of $f(x)$ at 0. Notice that the Fourier series of this odd function involves only sine functions; similarly the Fourier series of an even function involves on cosine terms (counting the constant term $a_0 \cos 0x$ as a cosine term).

Using Fourier series one can evaluate certain sums that would be difficult or impossible to realize as Taylor series. Perhaps the simplest and oldest example is the 2-series $\sum_{n=1}^{\infty}(1/n^2)$, which converges to $\pi^2/6$; this series is also a Taylor series, but of a function given by an integral which is impossible to evaluate by a closed formula. Looking instead at the Fourier series of $|x|$, we can deduce that $\sum_{n=1}^{\infty}(1/n^2) = \pi^2/6$. If we try to look at the next case $\sum_{n=1}^{\infty}(1/n^3)$, we find that Fourier series come close to being able to evaluate this sum, but can't quite do it; the best one can do is show that the alternating sum $\sum_{n=0}^{\infty}(-1)^n/(2n+1)^3$ of the odd reciprocals equal $\pi^3/32$ (by looking at the Fourier series of the function $g(x)$ defined to be x^2 for $x \in [0, \pi]$ and $-x^2$ for $x \in [-\pi, 0]$). It was shown only in 1979 that the $\sum_{n=1}^{\infty}(1/n^3)$, called *Apéry's constant*, is irrational; by now it is also known that infinitely many other sums $\sum_{n=1}^{\infty} 1/n^{2k+1}$ are also irrational.

Fourier series belong to a branch of mathematics called *harmonic analysis*; the idea in physical terms it to take the graph of a periodic function $f(x)$, regard it as the oscilloscope reading of a sound wave, and then decompose that wave into its constituent frequencies. This is where the harmonic series got its name.