Lecture 1-23

Starting from the series $\ln(1-x) = \sum_{n=0}^{\infty} -x^{n+1}/(n+1)$ and the series obtained form this by replacing x by $-x$, namely $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^{n+1}x^{n+1}/(n+1)$, both valid for $|x| < 1$, we get the series for $\ln \frac{1+x}{1-x} = 2\sum_{n=0}^{\infty} x^{2n+1}/(2n+1)$, again valid for $|x| < 1$. This is useful because any $y > 0$ can be written as $\frac{1+x}{1-x}$ for some $x \in [0,1)$, so that we now have a series converging to $\ln y$ for any $y \in \mathbb{R}^+$.

Another very useful series results from generalizing the familiar binomial theorem. Let $\alpha \in \mathbb{R}$ and form the *binomial series* $1 + \sum_{n=1}^{\infty}(\alpha(\alpha-1)\ldots(\alpha-n+1)/n!)x^n$; notice that the coefficient of x^n here is just $\binom{\alpha}{n}$ if α happens to be a nonnegative integer, but makes sense for any α. The ratio test shows that this series has radius of convergence 1 (unless α happens to be a nonnegative integer, in which case the series is a finite sum and always converges). Using the formula $(\sum_{n=0}^{\infty} a_n x^n)(\sum_{m=0}^{\infty} b_m x^m) = \sum_{n=0}^{\infty} c_n x^n$, for multiplying two power series, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$, we find that the sum $f(x)$ of the binomial series satisfies $(1+x)f'(x) = \alpha f(x), f(0) = 1$, whence we must have $f(x) = (1+x)^\alpha$ for $|x| < 1$. This formula turns out also to hold at $x = \pm 1$, provided that $\alpha > 0$. Starting from the resulting formula $1 + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2^n n!}x^{2n}$ for $(1-x^2)^{-1/2}$ and integrating term by term, we get the series $x + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2^n(2n+1)n!}x^{2n+1}$ for $\arcsin x$, which converges at both endpoints $x = \pm 1$ as well. Curiously there is no convenient formula for the coefficients in the Taylor series for $\tan x$ (in stark contrast to $\arctan x$), except in terms of a family of numbers called *Bernoulli numbers*, which were originally defined in a totally different context.

To compute Taylor series of given functions $f(x)$ at given points $x = a$ we have seen that we must evaluate the derivatives $f^{(n)}(a)$. Usually we do this by computing the first few derivatives and looking for a pattern. For example, if $g(x) = x^2 \ln x$ and $a = 1$ (Example 2 in the text, p. 614), we find that $g(x) = x^2 \ln x, g'(x) = x + 2x\ln x, g''(x) = 3 + 2\ln x, g^{(n)}(x) = (-1)^{n-1}2(n-2)!x^{-(n-2)}$, whence $g(1) = 0, g'(1) = 1, g''(1) = 3$, and $g^{(k)}(1) = (-1)^{k+1}2(k-3)!$ for $k \geq 3$. Hence the Taylor series for $g(x)$ at $x = 1$ is $(x-1) + (3/2)(x-1)^2 + 2\sum_{k=3}^{\infty} \frac{-1)^{k+1}2(k-3)!}{k!}(x-i)^k$. It has radius of convergence 1, converging at $x = 2$ but not at $x = 0$. It is often difficult or impossible to compute the sequence of derivatives $f(a), f'(a), \ldots$ explicitly; a useful technique for producing new power series from old ones is changing the variable. For example, since $e^x = \sum_{k=0}^{\infty} x^k/k!$ for all x we have $e^{-y^2} = \sum_{k=0}^{\infty}(-1)ky^{2k}/k!$ for all y (replace x by $-y^2$ in the first series; this is much easier than computing the higher derivatives of e^{-y^2} at 0). Similarly, but much more easily, we have $e^{-x} = \sum_{n=0}^{\infty}(-1)^n x^n/n!$ One can also multiply two power series $\sum a_n x^n, \sum b_n x^n$, using a formula given earlier; we have $\sum_{n=0}^{\infty} a_n x^n)(\sum_{m=0}^{\infty} b_m x^m) = \sum_{r=0}^{\infty} c_r x^r$, where $c_r = \sum_{n=0}^{r} a_n b_{r-n}$. Note that while both series have infinitely many terms, only finitely many of them contribute to the coefficient of any fixed power of x in the product, so the product always makes sense.

Having seen that products of power series can always be defined, we can now see that given any power (or Taylor) series $\sum a_n x^n$ whose constant term a_0 is 0, we can define its multiplicative inverse $\sum b_n x^n$, using the formula $a_0 b_0 = 1$ to solve for b_0, then the formula $a_0 b_1 + a_1 b_0 = 0$ to solve for b_1, and so on. For example, if you were to forget the formula

$1/(1-x) = \sum_{n=0}^{\infty} x^n$, you could follow the above recipe to find a multiplicative inverse for $1-x$, obtaining $\sum_{n=0}^{\infty} x^n$. It is even possible to plug a power series with 0 constant term into another power series, though you cannot expect to be able to write a formula for the nth term of a such a composite. For example, the power series for $\sin(\sin x)$ is $(x - x^3/3! + x^5/5! - \dots) - (1/3!)(x - x^3/3! + x^5/5! - \dots)^3 \dots = x - (1/3)x^3 + \dots$; there is no formula for the coefficient of x^{2n+1} in this last series, but one can at least see that only finitely many terms of the series contribute to it for fixed n, so that this coefficient is well defined. Also only odd powers of x appear in the series, since $\sin(\sin x)$ is an odd function. On the other hand, it would not be feasible to write down a formula for $\sin(\cos x)$ in quite the same way, since in collecting terms in $(1 - x^2/2! + x^4/4! - \dots) - (1/3!)(1 - x^2/2! + x^4/4! - \dots) + \dots$, we find that we must sum a power series even to compute the overall constant term. (We know in principle at least though that this can be done, as the function $f(x) = \sin(\cos x)$ is certainly at least infinitely differentiable at every point; it is not difficult to show that it is analytic everywhere too).

You will have noticed an obvious similarity in the coefficients of the Taylor series for $e^x, \sin x$, and $\cos x$ at $x = 0$. This is not an accident. Recalling the set $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ of complex numbers, in which we add and subtract two numbers $a + bi, c + di$ in the "obvious" way, obtaining respectively $(a + c + (b + d)i, (a - c) + (b - d)i$, and multiply using the relation $i^2 = -1$ and the distributive law, so that $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$, the power series $\sum_{n=0}^{\infty} z^n/n!$ converges for any $z \in \mathbb{C}$. Bearing in mind that the powers of i in $\mathbb{C}$ cycle among $1, i, -1, -i$, we see that $e^{ix} = \sum_{n=0}^{\infty} x^{2n}/(2n)! + i(\sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)! = \cos x + i \sin x$ for any $x \in \mathbb{R}$ (and in fact for $x \in \mathbb{C}$ as well): this expresses the fundamental relationship between the exponential and trigonometric functions that I alluded to in the context of differential equations last quarter. A nice exercise in multiplying power series is to verify that $e^x e^y = e^{x+y}$; since this holds for any $x, y \in \mathbb{C}$, not just $x, y \in \mathbb{R}$, we are led to the formula $e^{a+bi} = e^a(\cos b + i \sin b)$, which shows how to exponentiate any complex number.