Lecture 1-22

Continuing from last time, suppose that we have a Taylor series $\sum a_n(x-a)^n$ such that $|a_{n+1}/a_n| \to 1/R$ for some $R > 0$, so that the series converges for $|x-a| < R$. We have already seen the sum $f(x)$ of the series is continuous on $(a-R, a+R)$. We now observe that the ratio $|(n+1)a_{n+1}/na_n|$ of successive coefficients in the term-by-term differentiated series $\sum na_{n-1}(x-a)^{n-1}$ has the same limit as the corresponding ratio $|a_n/a_{n-1}|$ for the original series, whence the differentiated series has the same radius of convergence as the original one. We saw at the end of last time that the integrals of the partial sums of the differentiated series, normalized to take the value 0 at $x = a$, converge to the original series, so it is the integral of the continuous function that is the sum of the differentiated series. In particular, by the Fundamental Theorem of Calculus, $f(x)$ *is differentiable with derivative* $f'(x)$ *equal to the sum* $\sum na_{n-1}(x-a)^{n-1}$. Similarly, the term-by-term integrated series $\sum a_n(x-a)^{n+1}/(n+1)$ converges to the integral $\int_a^x f(t)\,dt$ for $x \in (a-R, a+R)$.

Thus starting from the geometric series $\sum_{n=0}^{\infty} x^n$, which we know has sum $1/(1-x)$ for $|x| < 1$ we get the series $\sum_{n=0}^{\infty} x^{n+1}/(n+1)$ for its integral $-\ln(1-x)$, or equivalently the series $-\sum_{n=0}^{\infty} x^{n+1}/(n+1)$ for $\ln(1-x)$, valid for $|x| < 1$. To work out the sums of the three series $\sum_{n=0}^{\infty} x^n/n!, \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$, and $\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$ we need a slightly more roundabout argument. The sum $f(x)$ of the first series satisfies $f(x) = f'(x)$, since the series for $f(x)$ is the same as that for $f(x)$; we also have $f(0) = 1$. We know from our work on differential equations that the only function $f(x)$ with these properties is e^x, so we have $e^x = \sum_{n=0}^{\infty} x^n/n!$ for all $x \in \mathbb{R}$ (in fact even for all x in the complex numbers $\mathbb{C}$). Similarly, the function $g(x) = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$ is such that $g''(x) = -g(x), g(0) = 0, g'(0) = 1$. Again from our earlier work on differential equations, we know that the only function $g(x)$ with these properties is $\sin x$, so $\sin x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)!$ for all x. Similarly, $\cos x = \sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$ for all x. Also, starting from the geometric series $1/(1+x^2) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$, valid for $|x| < 1$, we get the series for $\arctan x = \sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$, also valid for $|x| < 1$.

You will have noticed that some of these series converge at one or both endpoints of the interval of convergence. For example, the series $\sum_{n=0}^{\infty} -x^{n+1}/(n+1)$ converges at $x = -1$ by the alternating series test, but what is its sum there? The obvious guess would be $\ln(1-(-1)) = \ln 2$, but is this correct? Indeed it is, by something called *Abel's Theorem*, which says that *if a Taylor series* $\sum(a_n(x-a)^n$ *converges at* $x = a+R$*, then this series converges uniformly on the entire interval* $[a, a+R]$ *(or* $[a+R, a]$ *, if* $R < 0$*).* To prove this one shows by the argument used to prove Dirichlet's test that the tail of this series $\sum_{n=m}^{\infty} a_n(x-a)^n$ goes to 0 uniformly on $[a, a+R]$, since this tail takes the form $\sum_{n=m}^{\infty} A_n B_n$, where $\sum A_n$ has bounded partial sums and the B_n are decreasing. Hence the sum $\sum_{n=0}(-1)^n x^n/(n+1)$ defines a continuous function on the interval $[0,1]$ whose value at $x = 1$ must be the limit of its values $\ln(1+x)$ as $x \to 1^-$. (Note also that since $-\ln(1-x)$ blows up at $x = 1$, we would expect the corresponding series to diverge, and it does; it is our old friend the harmonic series.) In a similar way, the convergent alternating series $\sum_{n=0}^{\infty}(-1)^n x^{2n+1}/(2n+1)$ and $\sum_{n=0}^{\infty}(-1)^{n+1}x^{2n+1}/(2n+1)$ have sums $\arctan 1 = \pi/4$ and $\arctan(-1) = -\pi/4$, respectively.

Now that we know that any convergent Taylor series $\sum a_n(x-a)^n$ necessarily defines a differentiable function on its interval of convergence, we can iterate the argument used to prove this to deduce that in fact the sum of a convergent Taylor series defines an infinitely differentiable function on its interval of convergence. We can also recover the coefficients a_n of a Taylor series $\sum a_n(x-a)^n$ whose sum $f(x)$ is known; differentiating this series term by term n times and plugging in $x = a$, we see that we must have $a_n = f^(n)(a)/n!$, the nth derivative of f at a (interpreting as usual the 0th derivative of a function as the function itself). In particular, *it is not possible for two distinct convergent Taylor series to define the same function*, though of course the Taylor series of a function $f(x)$ at $x = a$ will look quite different from the Taylor series of the same function at another point $x = b$. Thus it is possible for a power series and a Taylor series (at a point other than $a = 0$ to converge to the same function.

If a function $f(x)$ happens to equal the sum of its Taylor series $\sum_{n=0}^{\infty} f^{(n)}(a)/n!$ at $x = a$, then we say that the function $f(x)$ is *analytic at* $x = a$. Of course an analytic function at a point must in particular be infinitely differentiable there; but unfortunately many infinitely differentiable functions fail to be analytic. First of all, it turns out that the sequence of derivatives $f^{(0}(a), f^{(1)}(a), \ldots$ of an infinitely differentiable function at a point can be any sequence a_n of real numbers, so there is no guarantee that the Taylor series of the infinitely differentiable function $f(x)$ at $x = a$ even converges anywhere except at $x = a$. Secondly, and more subtly, it is possible for the Taylor series of one function at a point to converge to a different function. Consider an old bugaboo from last quarter, namely the function $f(x)$ defined as e^{-1/x^2} for $x \neq 0$, while $f(0) = 0$. Last quarter, we showed that the nth derivative $f^{(n)}(x)$ of f at x takes the form $p_n(1/x)e^{-1/x^2)}$ at any point $x \neq 0$, whence by L'Hopital's Rule we have $f^{(n)}(0) = 0$ for all n. Hence the Taylor series of $f(x)$ at $x = 0$ is the 0 series, even though $f(x) \neq 0$ for any $x \neq 0$. Taking the power series for e^x and replacing x throughout by $-1/x^2$ we derive a power series in $1/x$ (or equivalently a power series in *negative* powers of x) that converges to $f(x)$ at any $x \neq 0$; the very existence of such a series rules out the possibility of any (ordinary) power series converging to x.