Lecture 1-21

We turn now to the main series of interest to us, namely power series. These take the form $\sum a_n x^n$ (the summation can begin at any index, but in practice this index is usually 0 or 1). It turns out that by studying this series for arbitrary values of x, not just $x = 1$, and determining not just when it converges but what it converges to, yields information about the series $\sum a_n$ that could not have been obtained directly. (Similarly, in studying the area under the graph of $y = x^2$ between, say $x = 1$ and $x = 2$, it is more fruitful to study the more general question of what this area is between any two limits for x than to study just this one area directly.)

The first thing to observe is that the set of values for x for which $\sum a_n x^n$ converges exhibits strong regularities. More precisely, suppose that this series converges at $x = R$ for some real number R. In particular, we must then have $|a_n R^n| \to 0$ as $n \to \infty$. It then follows for any x with $|x| < |R|$ that the series $\sum |a_n x^n|$ has terms less than or equal to the corresponding terms of the convergent geometric series $\sum (|x|/R)^n$ for sufficiently large n, whence $\sum a_n x^n$ converges absolutely. In fact, we have something even better: if we fix $\alpha > 0$ with $\alpha < |R|$, then the "tail" $\sum_{n=m}^{\infty} |a_n x^n|$ has terms dominated by those of $\sum_{n=m}^{\infty} (\alpha/|R|)^n$ for sufficiently large m, whence given $\epsilon > 0$ there is an index N for which any partial sum of $\sum a_n x^n$ lies with ϵ of the full sum $\sum a_n x^n$, for all $x \in [-\alpha, \alpha]$ simultaneously. We express this situation by saying that $\sum a^n x^n$ converges to its limit *absolutely and uniformly for all* $x \in [-\alpha, \alpha]$. We will see some beneficial effects of this uniform convergence soon.

For now we observe as a consequence that given any power series $\sum a_n x^n$ either it converges for all $x \in \mathbb{R}$ (in which case we say that its *radius of convergence* is infinite), or only for $x = 0$ (in which case we say its radius of convergence is 0), or else there is a positive constant R such that the series converges absolutely for $|x| < R$ but diverges for $|x| > R$ (in which case we say its radius of convergence is R). In this last case we allow any behavior at the extreme values $x = \pm R$: the series might converge conditionally or diverge at either or both of these values and in any case the radius of convergence is unaffected. The radius of convergence can be easily computed from the ratio test: if the series $\sum a_n x^n$ has $|a_{n+1}/a_n| \to L$ as $n \to \infty$, then its radius of convergence R is $1/L$, where for this purpose we take $1/0 = \infty, 1\infty = 0$. (More generally, if $\limsup |a_n|^{1/n} = L$, then once again $R = 1/L$, with the above conventions.)

Thus in particular the radius of convergence of $\sum_{n=0}^{\infty} x^n/n!$ is infinite; here we note that by a standard convention for power series $0^0 = 0! = 1$. Similarly, the radii of convergence of both $\sum_{n=0}^{\infty} (-1)^n x^{2n+1}/(2n+1)!$ and $\sum_{n=0}^{\infty} (-1)^n x^{2n}/(2n)!$ are both infinite (taking ratios of successive *nonzero* terms rather than just successive terms in the latter two series, as mentioned previously). An intermediate case is that of the geometric series $\sum_{n=0}^{\infty} x^n$ and its term-by-term differentiated series $\sum_{n=1}^{\infty} nx^{n-1}$; both of these series have radius of convergence 1.

Very similar to power series are *Taylor series* $\sum_{n=0}^{\infty} a_n (x-a)^n$, where a is a constant. Any such series converges either for all x, or $x = a$ only, or else there is a positive number R such that the series converges absolutely for $|x - a| < R$ but diverges for $|x - a| > R$. We again call R the radius of convergence.

Given a power or Taylor series $\sum a_n x^n$ or $\sum a_n(x-a)^n$ with a positive radius of convergence R, we now ask what sort of function $f(x)$ such a series converges to for x in the interval of convergence. To answer this question we consider the more general situation of a sequence of functions $f_0(x), f_1(x), \ldots$ defined on some interval $[a, b]$. Such a sequence is said to converge *pointwise* on $[a, b]$ if the sequence $f_0(x), f_1(x), \ldots$ converges for all $x \in [a, b]$; it converges *uniformly* on $[a, b]$ to a function $f(x)$ if for every $\epsilon > 0$ there is an index N such that $|f_n(x) - f(x)| < \epsilon$ for any $n > N$ and *any* $x \in [a, b]$ (the same N has to work for all x). Now it turns out that the pointwise limit of continuous functions need not be continuous (consider the sequence of functions $f_n(x) = x^n$ on the unit interval $[0, 1]$, whose limit is the function $f(x)$ equal to 0 for $x \in [0, 1)$ but equal to 1 for $x = 1$), but the *uniform* limit of a sequence of continuous functions must be continuous. To prove this, let the sequence f_n converge uniformly to f on $[a, b]$. For $x \in [a, b]$ and $\epsilon > 0$, first choose an index N such that $|f(x) - f_n(x)| < \epsilon/3$ for $n \geq N$; then choose $\delta > 0$ such that $|f_N(x) - f_N(y)| < \epsilon/3$ whenever $x, y \in [a, b]$ and $|x - y| < \delta$. Then, under the same hypothesis on x and y, we have $|f(x) - f(y)| \leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| < \epsilon$, as desired. In particular, given say a Taylor series $\sum a_n(x-a)^n$ with radius of convergence $R > 0$, we have seen that its sum $f(x)$ is the uniform limit of continuous polynomial functions $\sum_{n=1}^{m} a_n(x-a)^n$ on any interval $[-\alpha, \alpha]$ strictly contained in $(-R, R)$, and so is continuous on $(-R, R)$.

Returning to the setting of a sequence of continuous functions f_n converging uniformly on an interval $[a, b]$ to a function $f(x)$, a further general fact is that the integrals $\int_a^b f_n(x)\,dx$ of the f_n converge to the integral $\int_a^b f(x)\,dx$ of $f(x)$; to see this, just note that, given $\epsilon > 0$ there is an index N such that $|f(x) - f_n(x)| < \epsilon/(b-a)$ for $n \geq N$, whence $\int_a^b |f(x) - f_n(x)|\,dx < \epsilon$, as desired. We will use this result to show next time that a convergent power or Taylor series defines a differentiable function within the interval of convergence.