Lecture 1-17

We begin with a remarkable application of Dirichlet's test for conditionally convergent series. Fixing $x \neq 2k\pi$ for any integer k and defining the sequence (a_n) via $a_n = \sin nx$, we recall from trigonometry that $\sin A \sin B = (1/2)(\cos(A-B) - \cos(A+B))$. Multiplying $\sum_{i=1}^n a_i$ by $\sin(x/2)$, we find that the sum telescopes and we get $\sum i = 1^n a_i = \frac{\cos(x/2) - \cos(2n+1)(x/2)}{\sin((1/2)x)}$, whence the partial sums of $\sum a_n$ are bounded (by $2/|\sin(x/2)))$ and $\sum i = 1^\infty \sin(ix)(1/i)$ converges for all x (including multiples of 2π, as it converges to 0 in that case). There is actually a very nice formula for the sum of this series; it turns out to be $(\pi - x)/2$ for $0 < x < \pi, 0$ for $x = 0$ and $(-\pi - x)/2$ for $-\pi < x < 0$. (Thus a convergent sum of infinitely many continuous functions, unlike the sum of finitely many continuous functions, need not be continuous.) More generally, the series $\sum_{i=1}^\infty a_i \sin(ix)$ converges for any x if the a_i are nonnegative and decrease to 0 as i goes to infinity. We thus obtain a large family of convergent series, called *Fourier series* (because of the appearance of the $\sin ix$ terms; general Fourier series would include $\cos(ix)$ terms as well). Such series are of fundamental importance in physics and have a rich mathematical theory as well. As with the integral $\int_0^\infty \sin x/x \, dx$, it is not difficult to show that the convergence of $\sum_{i=1}^\infty \sin(ix)/i$ is conditional for any x with $\sin x \neq 0$.

So far all of our tests for convergence of series have involved either comparisons with other series or specialized hypotheses on the series. It is convenient to have a convergence test that can be easily applied to general series without the need to compare them to other series. This is furnished by the *ratio test*: given a series $\sum a_n$ for which the ratio $|a_{n+1}/a_n| \to L$ for some L as $n \to \infty$, the series converges absolutely if $L < 1$ and diverges if $L > 1$. More generally, if $\limsup |a_{n+1}/a_n| < 1$, then $\sum a_n$ converges absolutely, while if $\liminf |a_{n+1}/a_n| > 1$, then $\sum a_n$ diverges. Indeed, if $L = \limsup |a_{n+1}/a_n| < 1$, then there is $\alpha \in (L, 1)$ such that $|a_{n+1}| \leq \alpha |a_n|$ for n sufficiently large, whence there is some M with $|a_n| \leq \alpha^{n-M} |a_M|$ for $n \geq M$ and $\sum |a_n|$ converges by comparison with a convergent geometric series. If instead $L' = \liminf |a_{n+1}/a_n| > 1$ then there is $\alpha \in (1, L)$ with $|a_{n+1}| \geq \alpha |a_n|$ for sufficiently large n, whence a_n does not even go to 0 as $n \to \infty$, showing that $\sum a_n$ and $\sum |a_n|$ both diverge. Unfortunately, if $\liminf |a_{n+!}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$, then nothing can be said: the p-series for $p = 1$ (i.e. the harmonic series) diverges while the p-series for $p = 2$ converges and both have $a_{n+1}/a_n \to 1$ as $n \to \infty$. This last case amounts to a hole in the ratio test; unfortunately this turns out to be a rather huge hole, so much so that a whole battery of tests was devised in the 19th century precisely to deal with the case $|a_{n+1}/a_n| \to 1$ in the ratio test.

As an example, the series $\sum_{n=1}^\infty n!/n^n$ has the ratio $((n+1)!/n!)(n^n/(n+1)^{n+1} = (n/(n+1))^n$, which approaches $1/e$ as n goes to infinity, whence this series converges. More generally, for fixed x, the series $\sum_{n=1}^\infty (n!/n^n) x^n$ converges if $x < e$ but diverges if $x > e$. The series $\sum_{n=0}^\infty x^n/n!$ has a better convergence behavior, converging absolutely for all x. The series $\sum_{n=0}^\infty n x^{n-1}$ converges absolutely for $|x| < 1$.

The *root test* is very similar to the ratio test. A series $\sum a_n$ converges absolutely if $\limsup |a_n|^{1/n} < 1$ and diverges if $\liminf |a_n|^{1/n} > 1$. The proof is essentially the same as that of the ratio test. Once again the case $\liminf |a_n|^{1/n} \leq 1 \leq \limsup |a_n|^{1/n}$ is the one for which the test fails: there are both convergent and divergent series with this behavior.

One should exercise some care in applying either the ratio or the root test. For example, given the series $\sum_{n=0}^{\infty}(-1)^n x^{2n}/(2n)!$ for $x \in \mathbb{R}$, one could rewrite it as $\sum_{n=0}^{\infty} a_n x^n$, where $a_n = 0$ if n is odd and $a_n = (-1)^m/(2m)!$ if $n = 2m$ is even. Indeed, one would have to rewrite the series in this way if one insists that all nonnegative powers of x appear in it (as is done with power series, to be discussed later). In this last form neither the ratio nor the root test applies to this series, since every other ratio of consecutive terms is undefined and every other root of a term is 0. But, of course, if we return to the series in its original form, the ratio of the absolute values of two successive terms is $x^2/((2n+1)(2n+2)$, which for any $x \in \mathbb{R}$ goes to 0 as n goes to infinity; thus the series converges absolutely for all x.

Since Julie showed yesterday that a conditionally convergent series can be rearranged to converge to any real number, we should take a moment to show that this kind of nonsense cannot occur for absolutely convergent series: *if $\sum_{n=1}^{\infty} a_n$ converges absolutely to L, then any rearrangement of this series also converges to L.* Indeed, given $\epsilon > 0$, there is some N such that the sum $\sum_{i=N+1}^{\infty} |a_i|$ of the absolute values of the terms in this series is less than ϵ. Given any rearrangment $\sum b_n$ of $\sum a_n$, there is an index M such that the terms $a_1, \ldots, a_N$ all appear among $b_1, \ldots, b_M$. The sum of the other a_i appearing among the $b_1, \ldots, b_M$ and of the b_i for $i > M$ is bounded in absolute value by $\sum_{i=N+1}^{\infty} |a_i| < \epsilon$, so the b_i also sum to L, as desired. In a similar way one can show that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ both converge absolutely, say to L and M, respectively, then the product series $\sum_{n=0}^{N} c_n$ converges absolutely to LM, where $c_n = \sum_{i=0}^{n} a_i b_{n-i}$.