Lecture 1-15

Returning to and generalizing an example from an earlier lecture, we consider again series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms that are decreasing. The trick that we used to show that $\sum 1/n$ diverges while $\sum 1/n^2$ converges can be generalized to the *Cauchy condensation test*, which says that a series $\sum_{n=1}^{\infty} a_n$ of decreasing nonnegative terms converges if and only if the series $\sum_{n=1}^{\infty} 2^n a_{2^n}$ converges. Indeed, the $2^n - 1$th partial sum of this series is bounded above by $\sum_{i=0}^{n-1} 2^i a_{2^i}$ but below by $\sum_{i=0}^{n-1} 2^i a_{2^{i+1}}$, so the partial sums of the first series are bounded if and only if the partial sums of the second one are. This is a kind of discrete version of the integral test.

What about series $\sum a_n$ whose terms are not nonnegative, or integrals $\int_a^{\infty} f(x)\,dx$ for which the integrand $f(x)$ is not always nonnegative? The first thing to be said is the convergence of another series or integral with nonnegative terms always implies the convergence of the given one. More precisely, any series $\sum a_n$ for which $\sum |a_n|$ converges also converges; we say that such a series *converges absolutely*. Similarly, any integral $\int_a^{\infty} f(x)\,dx$ for which $\int_a^{\infty} |f(x)|\,dx$ converges also converges; we say the first integral converges absolutely. To verify these assertions, it suffices to note in the series case that the series $\sum b_n = -\sum(a_n + |a_n|$ and $\sum c_n = \sum |a_n|$ both have nonnegative terms and converge, the former by comparison with the series $2\sum |a_n|$. Hence $\sum a_n$ itself, which is the difference of these two series, also converges. Similarly, if $\int_a^{\infty} |f(x)|\,dx$ converges, then so does $\int_a^{\infty}(f(x)+|f(x)|)\,dx$, whence so does $\int_a^{\infty} f(x)\,dx$. In particular, the series $\sum \sin n/n^2$ and the integral $\int_1^{\infty}(\sin x/x^2)\,dx$ both converge absolutely, by comparing to $\sum 1/n^2$ and $\int_1^{\infty}(1/x^2)\,dx$, and so converge. There are however examples of series $\sum a_n$ and integrals $\int_a^{\infty} f(x)\,dx$ that converge even though $\sum |a_n|$ and $\int_a^{\infty} |f(x)|\,dx$ diverge; we say that such sums or integrals converge *conditionally*. The convergence depends on delicate cancellation between positive and negative terms (or values of the integrand) and so typically is very slow.

As an example, consider the *alternating harmonic series* $\sum_{n=1}^{\infty}(-1)^{n-1}/n$. Here we observe that the odd partial sums $s_1, s_3, \ldots$ form a decreasing sequence while the even partial sums $s_2, s_4, \ldots$ form an increasing sequence. Moreover it is easy to check that every odd partial sum s_{2n-1} is greater than or equal to every even one s_{2m}. Hence the sequences $s_1, s_3, \ldots$ and $s_2, s_4, \ldots$ both converge, and moreover their limits are the same since $a_{2n+1} = s_{2n+1} - s_{2n} \to 0$ as $n \to \infty$. Hence $\sum_{n=1}^{\infty}(-1)^{n-1}/n$ converges. More generally, any alternating series $\sum(-1)^{n-1}a_n$ for which the a_n are decreasing, nonnegative, and approach 0 as $n \to \infty$ converges, regardless of how fast or slowly the a_n approach 0; this is the *alternating series test*. Similarly, given the integral $\int_1^{\infty} \sin x/x\,dx$, we can use integration by parts to rewrite it as $(-\cos x/x^2)|_1^{\infty} + \int_1^{\infty} \cos x/x^2\,dx$. The first difference has a finite limit and the second integral converges absolutely by comparison with $\int_1^{\infty} 1/x^2\,dx$, so $\int_1^{\infty} \sin x/x\,dx$ converges. But the integral $\int_1^{\infty} |(\sin x/x)|\,dx$ diverges, since for any positive integer k the integrand $|\sin x/x|$ is bounded below by $1/2x$ on the interval $[k\pi/2 - \pi/6, k\pi/2 + \pi/6]$, whence the partial integrals of this integral are bounded below by $2\pi/6$ times $2/\pi$ times the partial sums of the harmonic series and so do not have a limit. Here the general fact is that given any continuous function $f(x)$ whose integrals $\int_a^b f(x)\,dx$ are bounded as a function of b (they need not have a limit as $b \to \infty$) and another differentiable nonnegative function $g(x)$ decreasing to 0 as $x \to \infty$, the integral $\int_a^\infty f(x)g(x)\,dx$ converges, typically conditionally. Here there is no simple formula for the value of $\int_1^\infty \sin x/x\,dx$ but there is one for $\int_0^\infty \sin x/x\,dx = \pi/2$ (defining the integrand to have the value 1 at $x = 0$, since its limit is 1 as $x \to 0$). Remarkably enough, the integral $\int_0^\infty (\sin^2 x/x^2)\,dx$ is also equal to $\pi/2$. There is no cancellation in the latter integral, but the integrand has smaller absolute value than the first integrand; it turns out that these two effects exactly cancel each other out.

There is a summation analogue of this criterion for an improper integral to converge conditionally. To state it, we start with an analogue of integration by parts, called *summation by parts*. Given real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$, set $s_k = \sum_{i=1}^k a_i$ for $1 \le k \le n, s_0 = 0$. Then $\sum_{i=1}^n a_i b_i = \sum_{i=1}^n (s_i - s_{i-1}) b_i = s_n b_n + \sum_{i=1}^{n-1} s_i(b_i - b_{i+1})$. Similarly, fixing an index $m \le n$ and setting $t_k = \sum_{i+m}^k a_i, t_{m-1} = 0$, we have $\sum_{i=m}^n a_i b_i = \sum_{i=m}^n (t_i - t_{i-1}) b_i = t_n b_n + \sum_{i=m}^{n-1} t_i(b_i - b_{i+1})$. Now let a_n and b_n be sequences of real numbers such that the partial sums of the a_n are bounded (in absolute value), say by M (as before, they need not converge) while the b_n are decreasing and nonnegative with $b_n \to 0$; once again, it does not matter how fast the b_n approach 0. Then *Dirichlet's test* states that *the series* $\sum_{i=1}^\infty a_i b_i$ *converges*. To prove this, we show that the partial sums attached to this series form a Cauchy sequence. Fixing indices m, n with $m \le n$, we find that the difference $d_{m-1,n}$ between the $(m-1)$th and nth partial sums is $\sum_{i=m}^n a_i b_i = t_n b_n + \sum_{i=m}^{n-1} t_i(b_i - b_{i+1})$. Now the t_i are differences of two partial sums of the a_i and as such are bounded in absolute value by $2M$, while the $b_i - b_{i+1}$ with $m \le i \le n-1$ form a so-called telescoping series, with sum $b_m - b_n$. The upshot is that $d_{m-1,n}$ is bounded by $2M(b_m + b_n)$, which gets arbitrarily small for m, n sufficiently large. Hence the partial sums of $\sum a_i b_i$ indeed form a Cauchy sequence and $\sum a_i b_i$ converges, as claimed. In particular, we can vastly generalize the alternating series test: given any pattern of $2k$ signs $+$ or $-$ having the same number of $+$ and $-$ signs, repeat it periodically to form a sequence of numbers a_n, each either 1 or -1. Then given any sequence b_n of nonnegative numbers decreasing to 0 as n goes to infinity, the series $\sum a_n b_n$ converges.