**Lecture 1-14**

Continuing from last time, we now give a criterion for a general sequence $s_n$ to converge. The intuition behind this is quite simple; if the sequence converges, so that the $s_i$ are getting close to the limit $L$ as $i$ increases, then the $s_i$ must also be getting close to each other as $i$ increases. Accordingly, we say that the sequence $s_i$ is *Cauchy* if for every $\epsilon > 0$ there is $N$ such that for any indices $n, m \geq N$ we have $|s_n - s_m| < \epsilon$. Then *a sequence is Cauchy if and only if it converges*. To prove this, assume first that the sequence $s_i$ converges, say to $L$. Then given $\epsilon > 0$ there is $N$ such that whenever $n \geq N$ we have $|s_n - L| < \epsilon/2$, whence whenever $n, m \geq N$ we have $|s_n - s_m| \leq |s_n - L| + |L - s_m| < \epsilon$, as desired. Conversely, suppose that $s_i$ is Cauchy. In particular, there an index $M$ such that for any $n, m \geq M$ we have $|s_n - s_m| < 1$, whence $|s_n| \leq |s_M| + 1$ for $n \geq M$. It follows that $s_i$ is bounded (by $\max(|s_1|, \ldots, |s_M| + 1)$, whence $s_i$ has a finite limit superior, say $L$. We claim that $s_i$ converges to $L$. To prove this, let $\epsilon > 0$ and choose $N_1$ such that $|s_n - s_m| < \epsilon/2$ whenever $n, m \geq N_1$. Defining $t_i = \sup(s_i, s_{i+1}, \ldots)$ as we did last time, there is an index $M$ with $|t_i - L| < \epsilon/4$ for $i \geq M$, and then an index $N_2$ with $|s_{N_2} - t_{N_2}| < \epsilon/4$ and $|s_{N_2} - L| < \epsilon/2$. Finally, for $i \geq N = \max(N_1, N_2)$ we have $s_i - L| \leq |s_i - s_{N_2}| + |s_{N_2} - L| < \epsilon$, as desired. We can also say that a sequence $s_i$ converges if and only if its limits superior and inferior coincide; in general the difference between these two limits provides a precise measure of the extent to which the sequence fails to converge.

In the first week of class last quarter we constructed the real numbers as Dedekind cuts, that is, as certain sets of rational numbers. An alternative approach is to construct the real numbers as equivalence classes of Cauchy sequences of rational numbers. In more detail, we define a Cauchy sequence of rational numbers as above, restricting to rational numbers $\epsilon$ (since we have not yet constructed the real numbers). We say that two Cauchy sequences $a_n$ and $b_n$ are *equivalent* if the interleaved sequence $a_1, b_1, a_2, b_2, \ldots$ is Cauchy. Then, as indicated above, the real numbers may be identified with equivalence classes of Cauchy sequences. In this approach the fundamental completeness property of the real numbers is not that any set of them that is bounded above has a least upper bound, but rather that any Cauchy sequence of them converges. To define what is meant by a Cauchy sequence of Cauchy sequences, we need to say when two Cauchy sequences $a_n$ and $b_n$ are less than $r$ apart for some rational number $r$; this holds if and only if $|a_n - b_n| < r$ for sufficiently large $n$. Then to find the limit of a Cauchy sequence of (equivalence classes of) Cauchy sequences, one starts with the $N_1$th term of the first sequence, where the index $N_1$ is chosen so that any two terms of this sequence past the $N_1$th term are less than 1 apart. The next term is then the $N_2$th term of the second sequence, $N_2$ chosen so that any two terms of this sequence past the $N_2$th are less than 1/2 apart, and so on. One disadvantage of this approach (which you can probably see already) is that equivalence classes of sequences are more awkward to deal with, both notationally and conceptually, than sets of rational numbers; but one advantage is that the definitions of the arithmetic operations on Cauchy sequences are very straightforward and require no splitting into cases.

We now formally introduce improper integrals, which we have already seen last term. Given a function $f(x)$ defined on $[a, \infty)$ (and assumed bounded and continuous there for simplicity) we define its improper integral $\int_a^\infty f(x)\,dx$ to be $\lim_{b\to\infty}\int_a^b f(x)\,dx$, if the limit exists; if it does not, we say that this integral *diverges*. If in addition the integrand $f(x)$ is nonnegative, then there are only two possible behaviors: either the set of "partial integrals" $\int_a^b f(x)\,dx$ for all $b \geq a$ is bounded or it is not. If it is not, then we now allow ourselves to write $\int_a^\infty f(x)\,dx = \infty$; if it is, then the least upper bound of the integrals $\int_a^b f(x)\,dx$ is the value of $\int_a^\infty f(x)\,dx$. As with infinite series with nonnegative terms, we have a *comparison test*: if $f(x), g(x)$ are nonnegative functions with $f(x) \leq g(x)$ for all sufficiently large $x$, and if $\int_a^\infty g(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$, while if $\int_a^\infty f(x)\,dx$ diverges, then so does $\int_a^\infty g(x)\,dx$. For example, we checked directly last quarter that $\int_1^\infty x^r\,dx$ converges if and only if $r < -1$; it follows that $\int_1^\infty \sin^2 x/x^r\,dx$ converges for $r > 1$. We showed that $\int_0^1 x^r\,dx$ converges exactly for $r > -1$. It follows that $\int_0^1 x^r \cos^2 x$ converges exactly for $r > -1$ as well, since we have $1/2 < \cos^2 x \leq 1$ for $x$ sufficiently small. The limit comparison test that we saw earlier for series carries over to integrals; if $f(x), g(x)$ are nonnegative functions such that $f(x)/g(x) \to L$ as $x \to \infty$, where $L$ is finite and nonzero, then $\int_a^\infty f(x)\,dx$ converges if and only if $\int_a^\infty g(x)\,dx$ does.

Returning now to infinite series, their theory and that of improper integrals come together in a beautiful way in the integral test: *given a sequence $a_n$ such that there is a decreasing continuous function $f(x)$ with $f(n) = a_n$, the series $\sum_{n=1}^\infty a_n$ converges if and only if the integral $\int_1^\infty f(x)\,dx$ does.* This follows since a typical partial sum $\sum_{i=1}^n a_i$ of $\sum_{i=1}^\infty a_i$ is bounded above by $\int_1^{n+1} f(x)\,dx$, but below by $\int_1^n f(x)\,dx$, this sum being a lower sum for the first integral and including an upper sum for the second integral. Hence the partial integrals have a limit if and only if the series converges. In particular, as we saw before, $\sum_{i=1}^\infty 1/i^p$ converges if and only if $p > 1$; similarly $\sum_{i=2}^\infty (1/i \ln i)$ diverges. In general, if $(a_n)$ is any sequence satisfying the above conditions and we look at the difference $t_n = \sum_{i=1}^n a_i - \int_1^n f(x)\,dx$ we find that the $t_n$ form a decreasing sequence of nonnegative numbers, which we know always has a limit. In the special case $f(x) = 1/x, a_n = 1/n$ this limit is called the *Euler-Mascheroni constant* and is usually denoted $\gamma$. Appallingly little is known about it; for example, we still do not know whether it is rational or irrational.