Lecture 1-13

We turn now to sequences and series. We have already given the basic definitions and so will only briefly recall these before proceeding to the main results and examples. We already know what it means for a sequence s_n of real numbers to have the limit L; from now on the main sequences s_n we will consider are the sequences of *partial sums* $s_n = \sum_{i=1}^n a_i$ of *infinite series* $\sum_{i=1}^\infty a_i$ of real numbers a_i. We say that $\sum_{i=1}^\infty a_i$ *converges* to L (or has sum L) if the corresponding sequence s_n converges to L. We say that $\sum_{i=1}^\infty a_i$ *diverges* if it does not converge. Last term we did not allow functions or sequences to have infinite limits, but here we relax this rule, allowing ourselves occasionally to write $\sum_{i=1}^\infty a_i = \infty$ if for every $x \in \mathbb{R}$ there is an index N such that $\sum_{i=1}^n a_i > x$ if $n \geq N$. We also allow infinite series to start at any index i, not just $i = 1$.

In most cases the sequence s_n of partial sums of an infinite series $\sum_{i=1}^\infty$ is much too complicated to study directly. One of the rare exceptions occurs for a *geometric series* $\sum_{i=0}^\infty ar^i$, where a, r are real numbers. Leaving aside the trivial case $a = 0$, you will derive in HW an explicit formula for $s_n = \sum_{i=0}^n ar^i$ and use this formula to show that the series converges if and only if $|r| < 1$; you will also work out its sum in that case. In general, it will take a while before we are able to work out the sum of any nongeometric convergent series, but in the meantime there are some fairly powerful tools to decide whether a given series converges or diverges. We begin with a very simple observation: if $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1} = 0$, since both s_n and s_{n-1} have the same limit as $n \to \infty$. Thus any series $\sum a_n$ for which $a_n \not\to 0$ as $n \to \infty$ diverges.

We start by considering series $\sum a_i$ with all $a_i \geq 0$. In this case the sequence s_n of partial sums is increasing, so we know by the first week of last term that $\sum a_i$ converges if and only if the s_n are bounded. There is an easy sufficient condition for this, called the *comparison test*: if we have $a_i \leq b_i$ for all but finitely many indices i and if $\sum b_i$ converges, then so does $\sum a_i$. If on the other hand we have $a_i \geq b_i$ for all but finitely many i and if $\sum b_i$ *diverges*, then so does $\sum a_i$. Applying this test to the famous *harmonic series* $\sum_{i=1}^\infty (1/i)$, it turns out to be more convenient to study the partial sums s_n with $n = 2^m - 1$ one less than a power of 2 than in general. Setting $j = 2^m - 1$, we find that the partial sum s_j can be written as the sum of m subsums $\sum_{i=2^k-1}^{2^k}(1/i)$ (for $1 \leq k \leq m$) and the kth subsum is greater than or equal to $\sum_{i=2^k-1}^{2^k}(1/2^k) = 1/2$, whence even the set of partial sums s_{2^m-1} is not bounded and the harmonic series diverges. On the other hand, if p is a positive real number, then the *p-series* $\sum_{i=1}^\infty (1/i^p)$ is such that its $(2^m - 1)$th partial sum s_{2^m-1} is less than or equal to $\sum_{k=1}^m \sum_{i=2^k-1}^{2^k}(1/2^{k-1})^p = \sum_{k=1}^m 2^{k-1}/2^{p(k-1)}$, a partial sum of a convergent geometric series whenever $p > 1$. Hence the partial sums s_{2^m-1} of the p-series are uniformly bounded, whence the set of all partial sums s_n of this series is bounded (since any $s_n < s_{2^m-1}$ for sufficiently large m) and the *p-series converges exactly for* $p > 1$. This is our second family of series where we can pinpoint the dividing line between convergence and divergence precisely, the first one being the geometric series, and as with that series the transition from convergence to divergence takes place at the value 1. Note that the convergence of the p-series for $p > 1$ is usually established by the so-called integral test, but the above argument shows that the latter test is not necessary for this purpose.

With two families of series under our belt whose convergence behavior is known, we are in a position to test many other series for convergence using the comparison test. For example, the series $\sum_{n=1} \frac{n^2}{n^5+1}$ converges by comparison with the 3-series $\sum_{n=1}^{\infty}(1/n^3)$, since $\frac{n^2}{n^5+1} < \frac{1}{n^3}$ for all $n \geq 1$; similarly $\sum_{n=1} \frac{\sqrt{n}+4}{2n}$ diverges by comparison with the 1/2-series. What about the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$? We cannot compare directly with the harmonic series, since the inequalities go the wrong way; that is, we have $\frac{n}{n^2+1} < \frac{n}{n^2}$ rather than $\frac{n}{n^2+1} > \frac{n}{n^2}$. If we compute the limit of the ratio $\frac{n/(n^2+1)}{(1/n)}$ as $n \to \infty$, however, we find that this limit is 1, whence we have (say) $\frac{n}{n^2+1} > \frac{1}{2n}$ for sufficiently large n and $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by comparison with the harmonic series times 1/2. The general principle at work here is called the *limit comparison test*: given two series $\sum a_n, \sum b_n$ with nonnegative terms such that the ratio a_n/b_n approaches a finite nonzero limit L as $n \to \infty$, these series converge or diverge together (that is, one converges if and only if the other does). This follows since for sufficiently large n we have both $a_n < 2Lb_n$ and $a_n > (1/2)Lb_n$. (Note also that if the ratio a_n/b_n has limit ∞ and $\sum b_n$ diverges, then so does $\sum a_n$; if the ratio a_n/b_n has the limit 0 and $\sum b_n$ converges, then so does $\sum a_n$.) The limit comparison test vastly extends the scope of the (ordinary) comparison test, since it is often hard to check that $a_n < b_n$ for all but finitely many n, but easy to evaluate $\lim a_n/b_n$.

Before going on with series, we return to sequences, deriving a general necessary and sufficient condition for a sequence to converge. We already know that an increasing or decreasing sequence converges if and only if it is bounded (above and below), but what about general sequences? To study these we introduce two kinds of limits that are defined for all sequences, convergent or not. Given a sequence s_n that is not bounded above (as a set), we say (by definition) that its *limit superior* is infinite and we write $\limsup s_n = \infty$. Otherwise the set $S_i = \{s_i, s_{i+1}, \dots\}$ is bounded above for all i and so has a least upper bound t_i; moreover we have $t_i \geq t_{i+1}$ since $S_i \subset S_{i+1}$. If the sequence t_i is not bounded below, we write $\limsup s_i = -\infty$; otherwise, this sequence has a limit (equal to its greatest lower bound), which we denote by $\limsup s_i$. We define $\liminf s_i$ in a similar manner, working with the greatest lower bounds of the S_i if they exist and writing $\liminf s_i = -\infty$ if they do not. Next time we will give a criterion for s_i to converge and use the limit superior to identify the limit of s_i whenever this sequence satisfies the criterion.