Lecture 1-10

We wrap up Chapter 10 with a brief discussion of surface areas of solids of revolution and centroids of parametrized curves. We begin with surface areas. We start with the graph of a parametrized curve $(x(t), y(t))$ lying in the upper half plane and defined on an interval $[a, b]$. We assume that the function $x(t)$ is strictly increasing on $[a, b]$; note that this assumption is not made explicit in the discussion on the text on p. 518, though it should have been. Now we rotate the region between this graph and the interval $[a, b]$ on the x-axis about this axis, obtaining a solid of revolution. The volume of this solid is $\pi \int_a^b y(t)^2 x'(t)\, dt$, by a straightforward extension of the formula for this volume in the special case of a graph of a function (this formula does not appear in the text). The surface area of this solid can be written as the limit of Riemann sums for the integral $\int_a^b 2\pi y(t)\sqrt{x'(t)^2 + y'(t)^2}\, dt$; this formula ultimately arises from the well-known formula πRs for the lateral surface area of a cone with base radius R and slant height s. In particular, if the graph of a positive function $y = f(x)$ on an interval $[a, b]$ is rotated about the x-axis, the surface area of the resulting solid of revolution is given by $2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx$. Thus for example a sphere of radius R is obtained by rotating the parametrized curve $(x, y) = (R\cos t, R\sin t)$ about the x-axis for $0 \le t \le \pi$, whence its surface area is $2\pi R^2 \int_0^\pi \sin t\, dt = 4\pi R^2$, as expected. Note that, unlike the corresponding volume computation, it makes no difference in the surface area computation that the graph is traced here in the "wrong" direction (from right to left instead of left to right), though of course we still have to make sure that every point on the circle is traced exactly once over the given interval.

An amusing example arises if the graph of $y = x^{-2/3}$ for $x \ge 1$ is rotated about the x-axis. The resulting solid has finite volume $\pi \int_1^\infty x^{-4/3}\, dx = -3x^{-1/3}|_1^\infty = 3$, but its surface area is infinite. Here we cannot actually evaluate the surface area integral, but we can observe that the integrand $2\pi x^{-2/3}\sqrt{1 + (4/9)x^{-10/3}}$ is bounded below by $2\pi x^{-2/3}$, whose integral from 1 to ∞ diverges. Thus "you can fill up this solid (called *Gabriel's horn*) with paint but you can't paint its sides".

Returning now to two dimensions, we consider centroids of parametrized curves. These are by definition certain weighted averages of coordinates of points on the curves; note that for closed curves they are *not* the same as centroids of regions enclosed by such curves. More precisely, one starts with the integrand $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$ for computing the arclength of a parametrized curve and then introduces multiplicative weights $x(t), y(t)$ to this integrand to compute the coordinates $(\bar{x}, \bar{y})$ of this centroid, so that $\bar{x} = (1/L)\int_a^b x(t)\, ds, \bar{y} = (1/L)\int_a^b y(t) ds$, where the curve is parametrized over the integral $[a, b]$ and has arclength L. Here we do not need to make any assumptions about the signs or monotonicity of $x(t)$ or $y(t)$. For example, the x-coordinate $\bar{x}$ of the centroid of the quarter-circle $\{(R\cos t, R\sin t) : 0 \le t \le \pi/2\}$ equals $(2/\pi R)\int R^2 \cos t\, dt = 2R/\pi$; by symmetry the y-coordinate $\bar{y}$ of this centroid is also $2R/\pi$. By contrast, the centroid of the disk enclosed by this quarter-circle and the positive x- and y-axes is $(4R/3\pi, 4R/3\pi)$; it is closer to the center of the circle than the previous centroid since it is a weighted average of points most of which are closer to the center than any point on the circle. In general, the centroid of a closed curve lies inside the region it encloses, but closer to its boundary than the centroid of this region.

There is a beautiful relationship between surface areas of solids of revolution and centroids of curves generating them. This is given by *Pappus's Theorem*, which asserts that *if a plane curve is rotated about an axis in its plane such that the axis does not meet the curve and perpendicular lines to the axis meet the curve in at most one point, then the surface area of the solid of revolution equals the distance from the centroid of the curve to the axis of rotation times the arclength of the curve.* This follows at once from the above formulas.

While we have defined the arclength of a parametrized curve segment to be the limit of the sums of lengths of line segments joining successive points on that curve segment as the number of such points approaches infinity and the distance between two successive points approaches 0, it turns out that we *cannot* define areas of surfaces as limits of the sums of areas of inscribed triangles. The problem is that triangles inscribed in a surface may be more nearly perpendicular than parallel to the surface, so that the sum of their areas can be arbitrarily large. A number of definitions of area for more general kinds of surfaces have been proposed that disagree with each other; fortunately all agree with the formula we have given above for solids of revolution.