

Home Work Problem Set VII

1. There live 9 happy and 9 unhappy princesses in the Land of Evens and Odds. Schmerlin the Magician has just learned three new spells. The first spell makes any two unhappy princesses of his choice happy. His second spell transforms any pair of happy princesses into unhappy ones. The third spell switches the moods of a happy and an unhappy princess: the happy one becomes unhappy, and the unhappy one turns happy. Schmerlin intends to make all the princesses happy. Prove that these three spells are not sufficient for making his dream come true:

(a) Observe how Schmerlin's spell changes the number of unhappy princesses: how many more or how many fewer unhappy princesses will there be after a single spell?

(b) Currently, the number of unhappy princesses is odd. Suppose that Schmerlin utters one of his spells. Prove that the number of unhappy princesses will remain odd.

(c) Suppose that Schmerlin performs several spells in a row. Prove that the number of unhappy princesses will remain odd.

(d) Is it possible for the number of unhappy princesses to eventually go down to zero? If yes, show how. If no, explain why.

2. Three types of magic fruit, the apples of wisdom, the pears of bravery, and the plums of kindness, grow on the Magic Tree in the center of Parity Kingdom. From time to time, some of these fruit are harvested for the benefit of the Kingdom. The Magic Tree immediately re-grows the picked fruit according to the following set of rules:

- If a single fruit is picked from the tree, another fruit of the same kind grows in its place.
- If 2 apples are picked, 4 pears grow back.
- If 2 pears are picked, 4 plums grow back.
- If 2 plums are picked, 4 apples grow back.
- If 2 fruit of different kind are picked, nothing happens.

Currently, the tree has 11 apples, 10 pears and 8 plums. The wicked witch plans to weaken the Kingdom by stealing all the fruit. She intends to sneak to the tree several mornings in a row and pick one or two fruit every time. Is there a way for her to pick all the fruit of the tree? Either show how or explain why not.