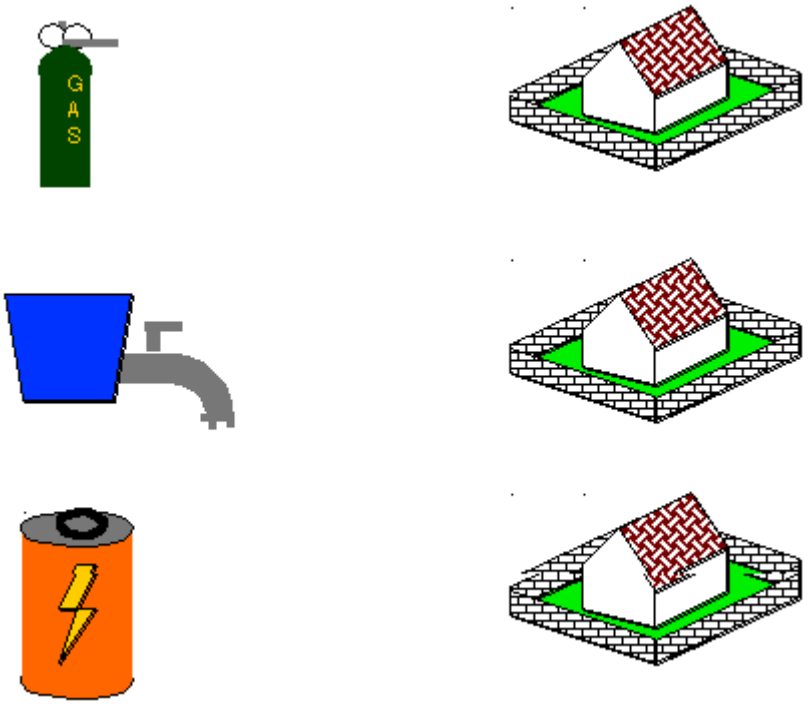
Montlake Math Challenge

October 23, 2008

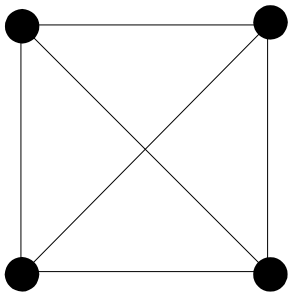
Graph Theory II

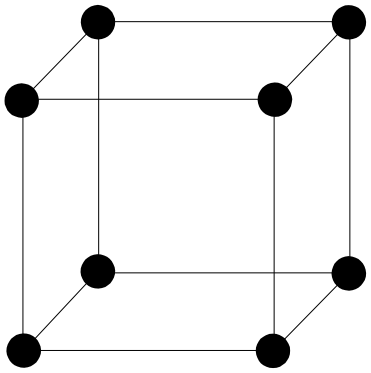
Problem 1: There are three houses on a street. Each house needs to be hooked up to a source for each of three utilities – water, electricity, and gas. Is it possible to do this so that none of the supply lines cross?



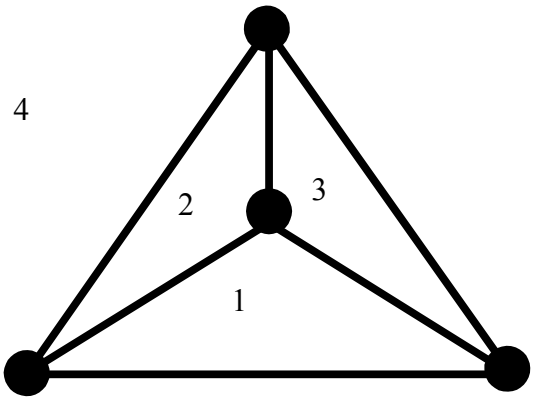
A graph is called **planar** if it can be drawn so that none of its edges cross one another.

Example 1: Both of these graphs are planar. Can draw them so that their edges do not cross?





For a planar graph, a **face** is a region enclosed by the edges of a graph. For technical reasons, we say that the area outside the graph is also a face, called the **infinite face**. For example, this graph has four nodes, six edges, and four faces, labeled 1,2,3,4.



Problem 2: Draw a planar graph of your own.

a) How many nodes does your graph have? Call this number N.

b) How many edges does your graph have? Call this number E.

c) How many faces does your graph have? Call this number F. Don't forget the infinite face!

d) What is N-E+F?

Problem 3: Answer questions (a) – (d) for the two graphs in Example 1.

Problem 4: Do you notice a pattern? If I give you a planar graph with N nodes, E edges, and F faces, what is N-E+F?

It is a fact that if a planar graph has N nodes and E edges, then

$$E \leq 3\mathrm{N} - 6.$$

Problem 5: For each of these three graphs, what are E and N? What is 3N-6? Check that $E \leq 3\mathrm{N} - 6$.

Problem 6: Let's go back to problem 1. Draw a graph that represents the water, gas, and electricity lines being hooked up to the houses.

a) How many nodes are there in the graph?

b) How many edges are there in the graph?

c) What is 3N-6?

d) Is this graph planar? Why or why not?

Problem 7: Decide if the following graphs are planar. If the graph is planar, draw it without edge crossings. If it is not planar, explain why not.