Small Boundaries and Circular Reasoning

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UW Math Hour, May 2024

UNIVERSITY of WASHINGTON

A story so old that even paintings about it are old

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Leonard Limosin, *Dido, Queen of Carthage*, 1564-1565

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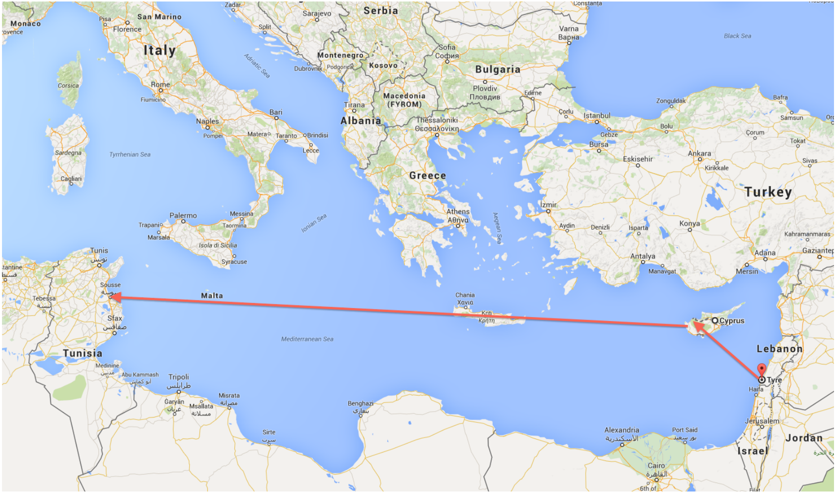
The word limousine is derived from the name of the French region Limousin; however, how the area's name was transferred to the car is uncertain. (wikipedia)

A story so old that even paintings about it are old

Guérin, *Dido and Aeneas*, painted 1815



800 BC, political difficulties in Tyre, Dido flees to Tunisia

There she meets King Iarbas and asks him for some land.

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(Pompeius Trogus, *Historiae Philippicae et Totius Mundi Origines et Terrae Situs*, first Century BC, as excerpted by Marcus Junianus Justinus Frontinus 300 years later)

Gregorio Lazzarini, Venice, $\sim$ 1700

At last they landed, where from far your eyes
May view the turrets of new Carthage rise;
There bought a space of ground, which Byrsa call'd,
From the bull's hide they first inclos'd, and wall'd.

(Vergil, *Aeneid*, written 29-19 BC)

Byrsa today!

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Queen Dido's great insight!

If you have a string of fixed length and
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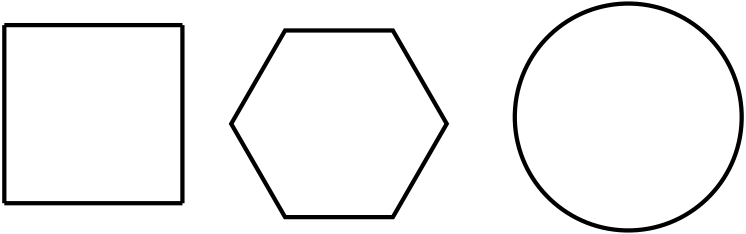

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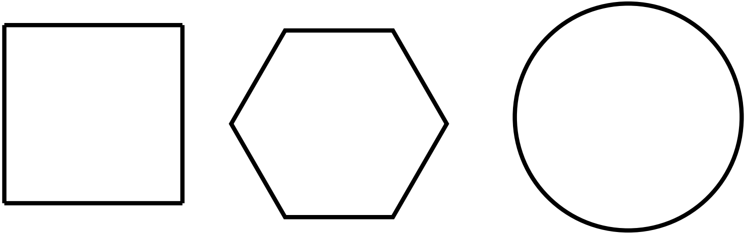

Or: among all sets in the plane with a fixed perimeter,
the circle has the largest area.

Civ 6 (2019)


SID MEIER'S
CIVILIZATION VI
GATHERING STORM
FIRST LOOK
PHOENICIA

Old World (2020)

Tunisian Dinar

Fun with language!

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Note that

*to surround (circumdare): 'giving (***dare***) a circle (***circus***)'*

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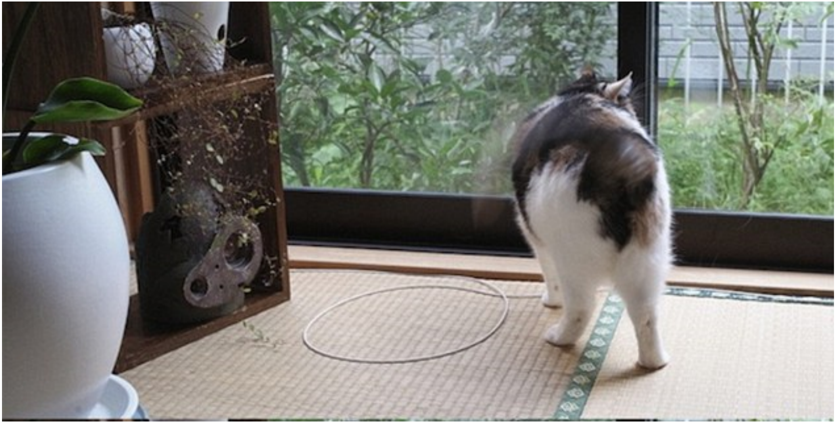
(Matthäus Meridan, Map of Paris, 1615)

Cat circles!



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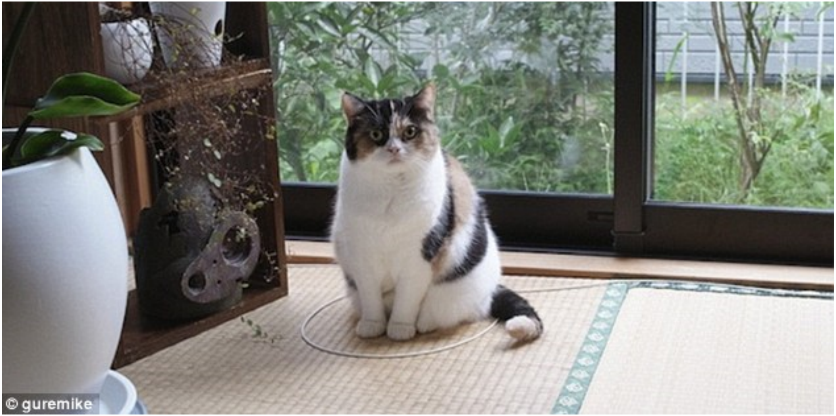
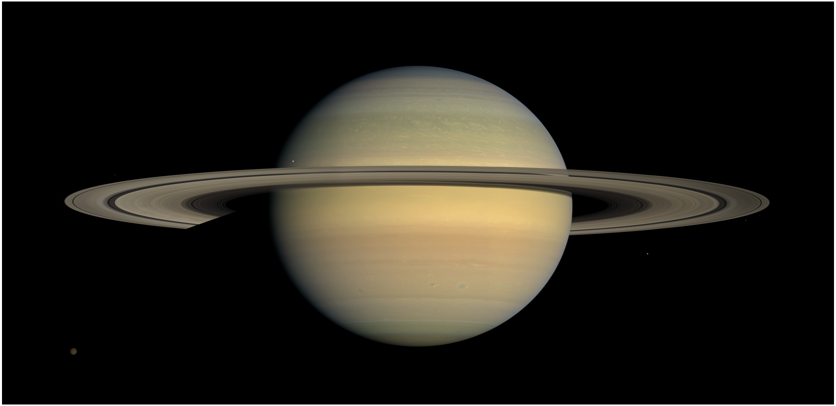
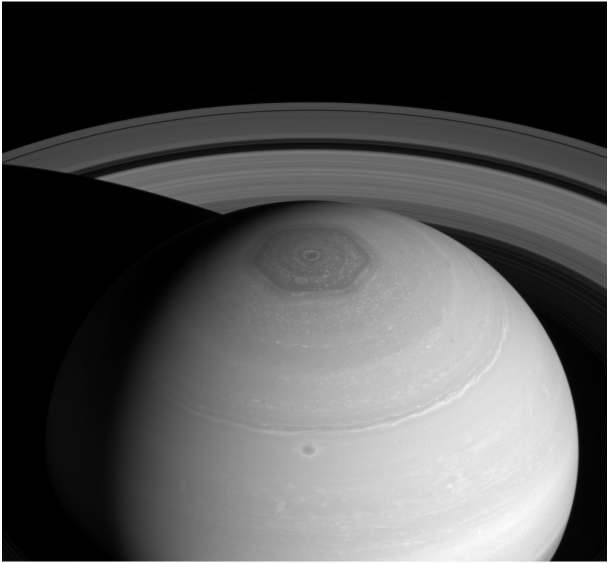


Cat circles!

Nature loves circles and balls



and hexagons!



The first rigorous understanding

Jakob Steiner (1796-1863)

Steiner symmetrization

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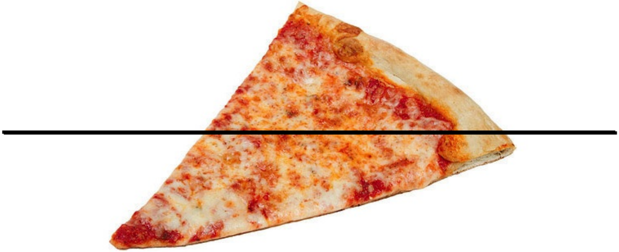

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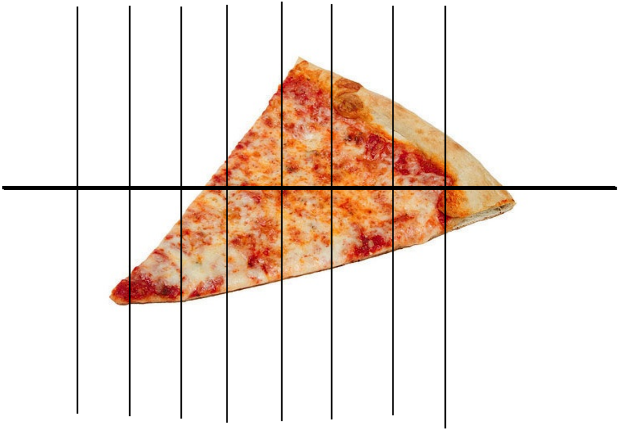

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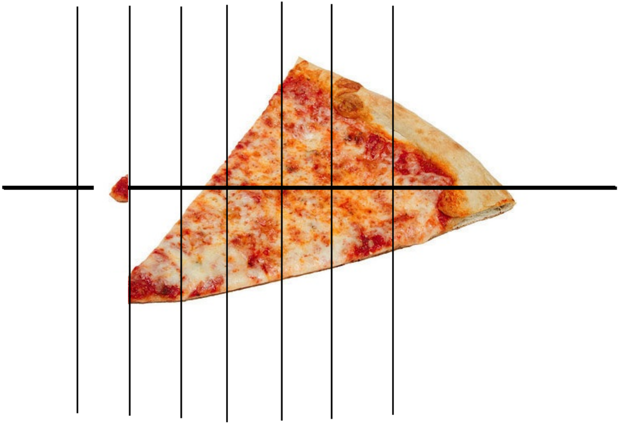

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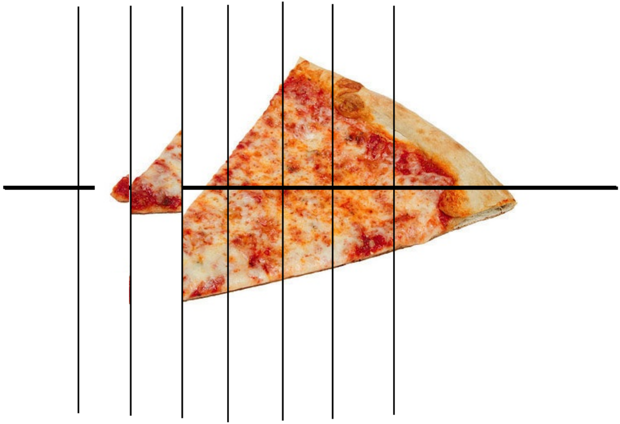

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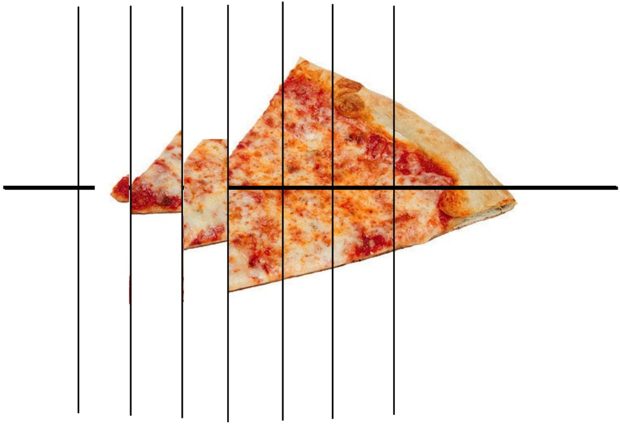

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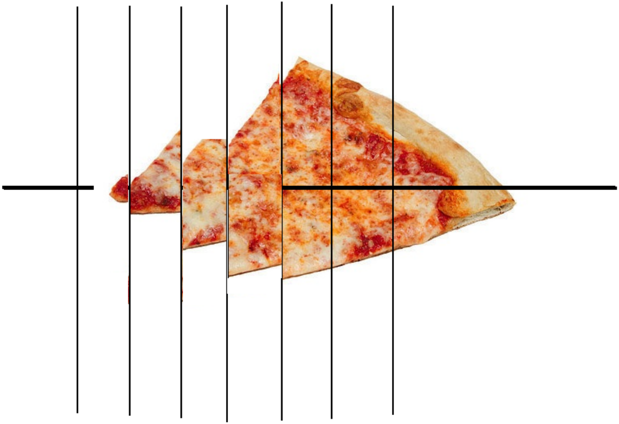

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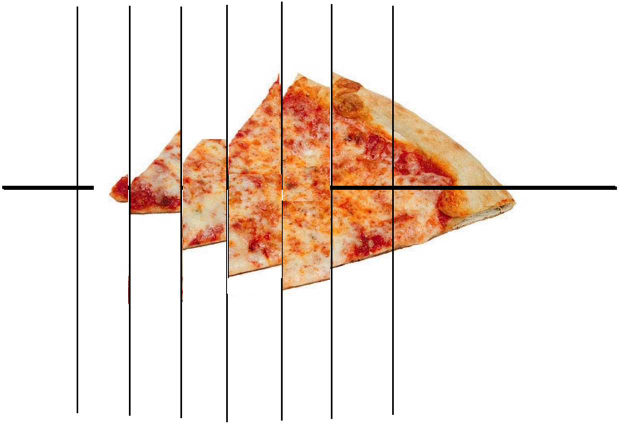

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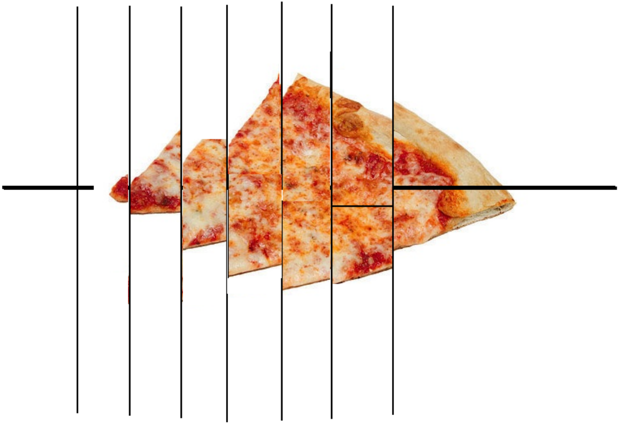

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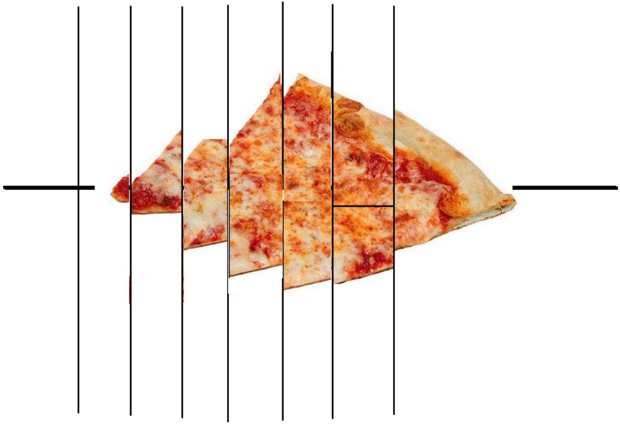

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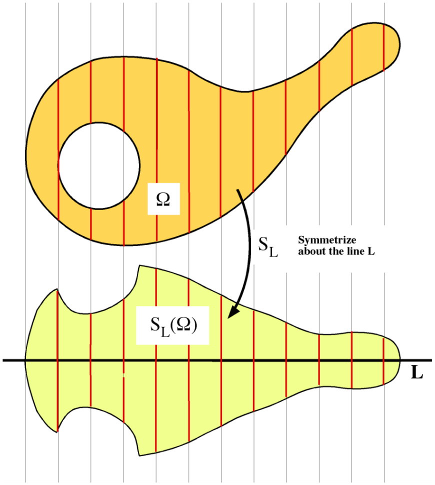

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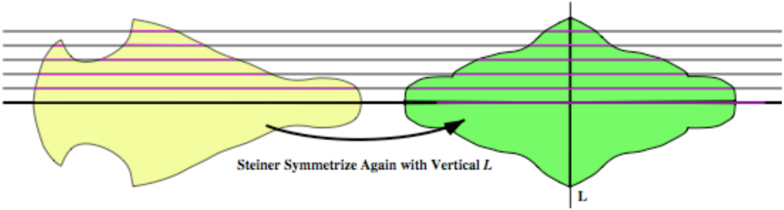

Theorem (Jakob Steiner)

This process keeps the area the same but makes the perimeter smaller (or leaves it the same).

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Is this a good reason?

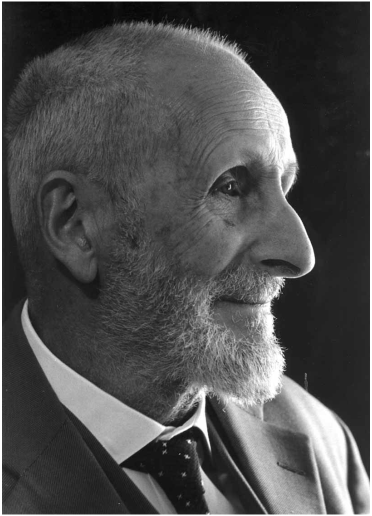
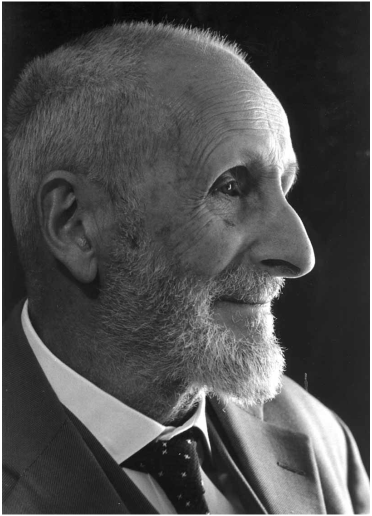


Figure: Oskar Perron (1880 - 1975)

If there is an optimal shape,
then it has to be a circle...



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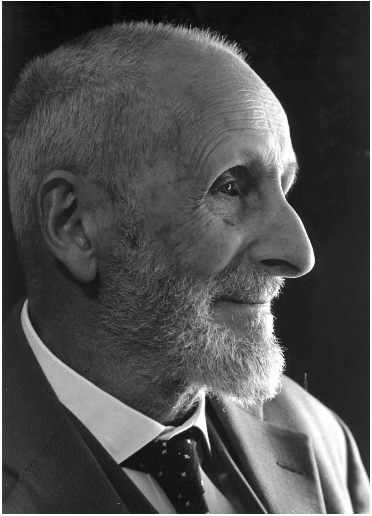
Perron's Theorem. $N = 1$ is the largest integer.



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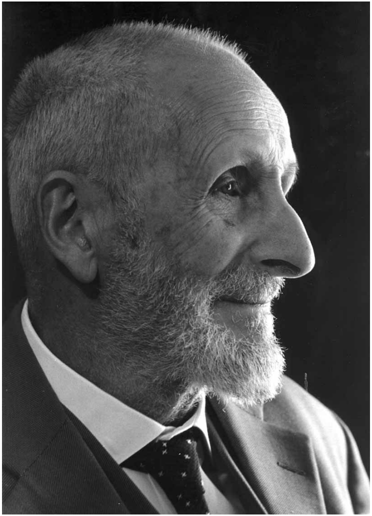

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Perron's Theorem. $N = 1$ is the largest integer.
Proof. $N^2 > N$ is always bigger unless $N = 1$.

...and maybe there is no optimal shape?

Figure: Wilhelm Blaschke (1885-1962)

Blaschke Selection Theorem

Any sequence of bounded convex domains has a convergent subsequence.

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which concludes the argument.

The notion of boundary

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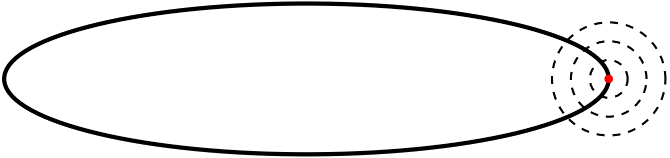

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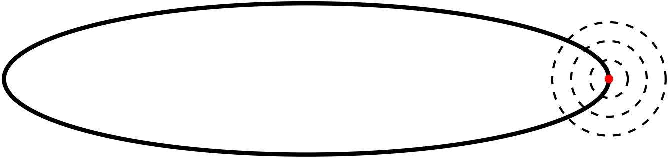

- A point $\mathbf{x} \in \mathbb{R}^n$ is called a **boundary point** of S if every ball centered at $\mathbf{x}$ contains both points in S and points in S^c. (Note that if $\mathbf{x}$ is a boundary point of S, $\mathbf{x}$ may belong to either S or S^c.) The set of all boundary points of S is called the **boundary** of S and is denoted by ∂S:

$$\partial S = \{\mathbf{x} \in \mathbb{R}^n : B(r, \mathbf{x}) \cap S \neq \varnothing \text{ and } B(r, \mathbf{x}) \cap S^c \neq \varnothing \text{ for every } r > 0\}.$$

(Folland, Advanced Calculus)

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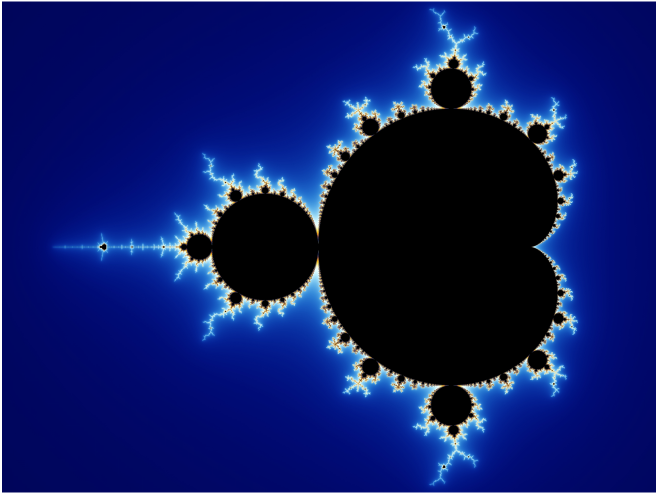

The Mandelbrot set

New challenges: data science!

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BOUNDARY ESTIMATION FROM POINT CLOUDS: ALGORITHMS, GUARANTEES AND APPLICATIONS

JEFF CALDER, SANGMIN PARK, AND DEJAN SLEPČEV

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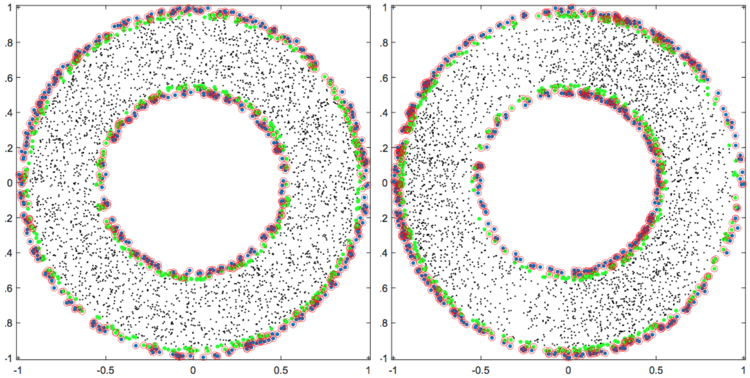
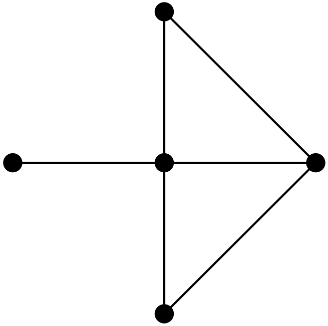
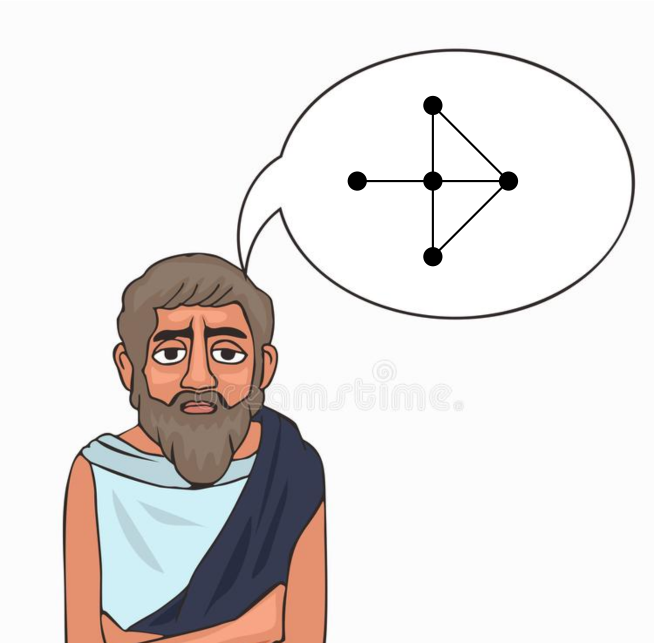


Can one define the boundary of a graph?



A graph is the collection of points (the *vertices*) some of which are connected (by *edges*) and others are not. Example: people on a social network and whether they are friends or not.

Graphs are thought of abstract ideals



Chartrand, Erwin, Johns, Zhang boundary

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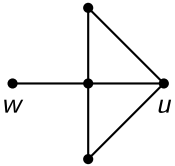


Figure: u is a boundary vertex because $d(w, u) = 2$ and every neighbor of u is at most distance 2 from w.

Chartrand, Erwin, Johns, Zhang boundary

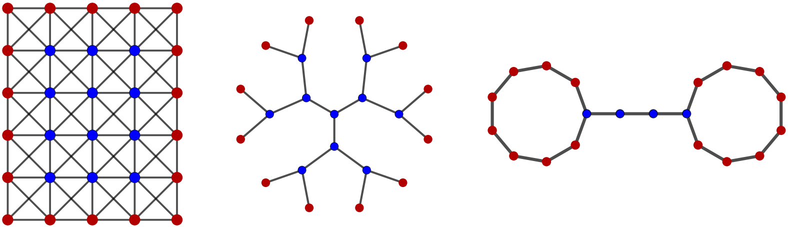
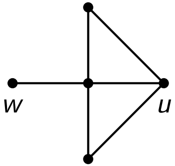


Figure: Graphs, their boundary $(\partial G)^*$ (red) and interior $V \setminus (\partial G)^*$ (blue).

The CEJZ boundary is pretty natural: if points end up being endpoints in the sense discussed, they should be in the boundary (whatever the definition of boundary ends up being).



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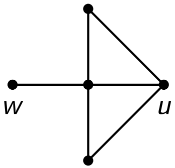

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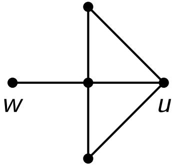

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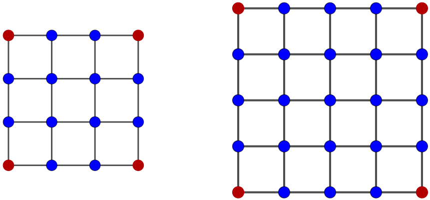

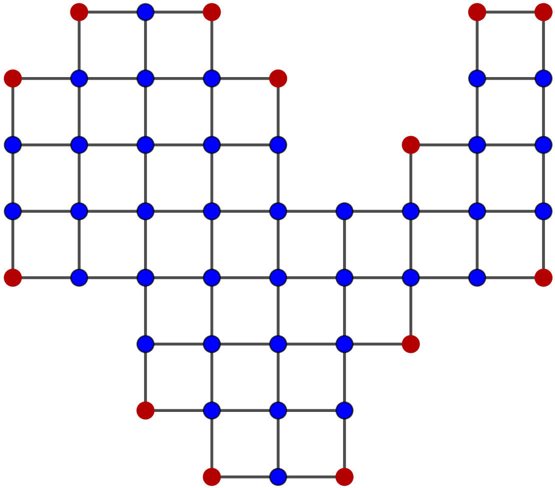

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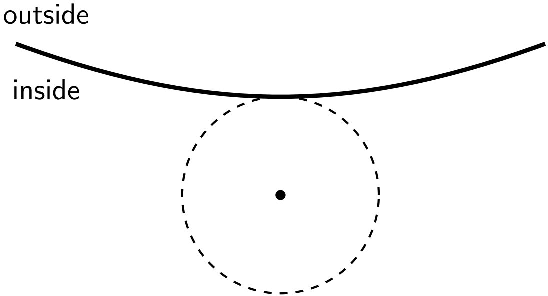

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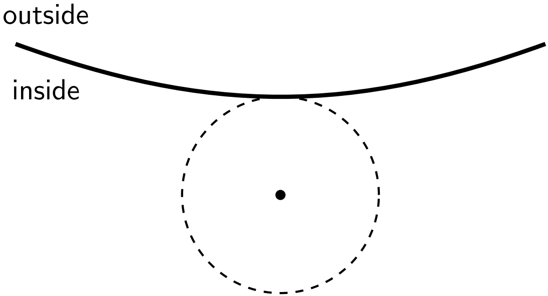

The boundary is not actually being classified as being part of the boundary since there are always points that are further away.

CEJZ boundary visualized!

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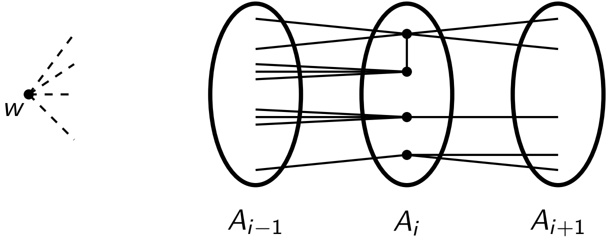

Figure: Vertices at distance $i-1, i, i+1$ from w.

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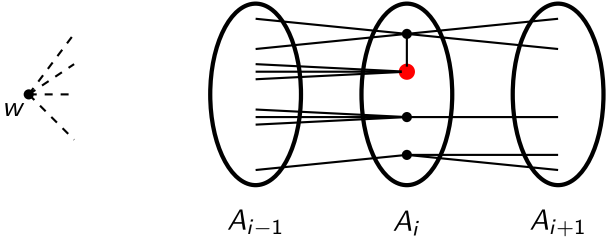

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A new type of boundary

We will relax the definition and instead consider vertices such that the *average* neighbor is closer to another vertex.

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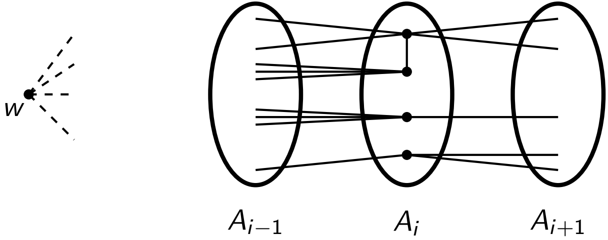

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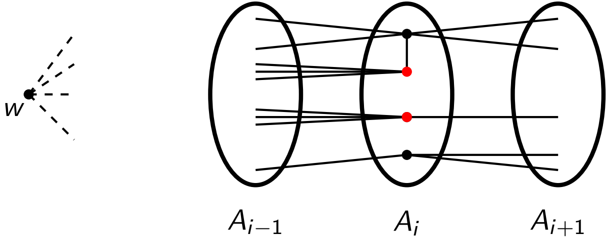

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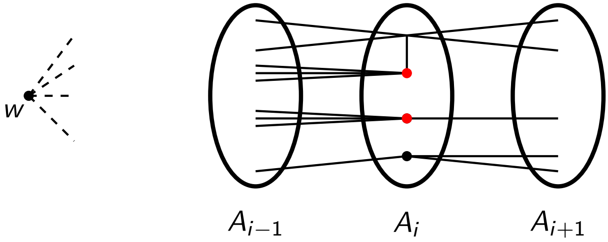

Formal definition is (don't read it)

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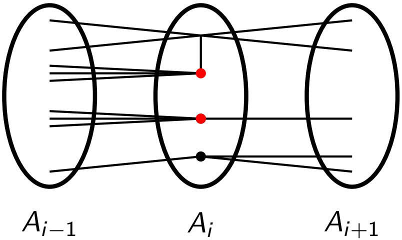


Formal definition is (don't read it)

$$\partial G = \left\{ u \in V \middle| \ \exists w \in V : \ \frac{1}{\deg(u)} \sum_{(u,v)\in E} d(w,v) < d(u,w) \right\}.$$

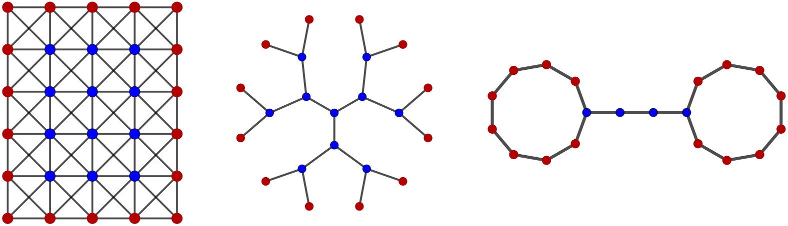


Figure: Graphs, their boundary ∂G (red) and interior $V \setminus \partial G$ (blue).

This boundary ∂G always contains the CEJZ boundary $(\partial G)^*$ and sometimes they coincide (as in these pictures).

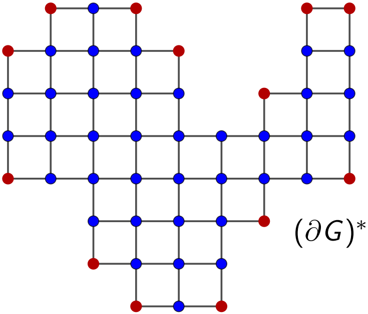
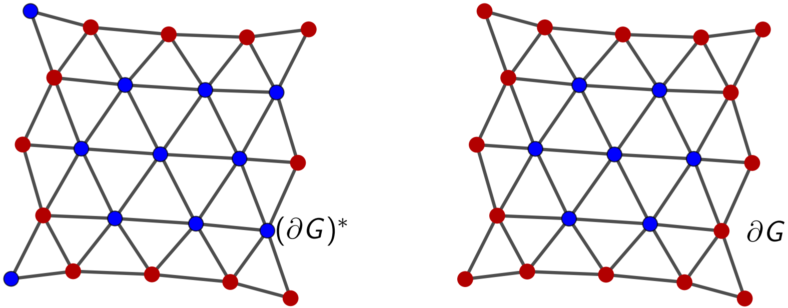


Figure: Left: Chartrand-Erwin-Johns-Zhang boundary $(\partial G)^*$ (in red). Right: our notion of boundary ∂G (in red).



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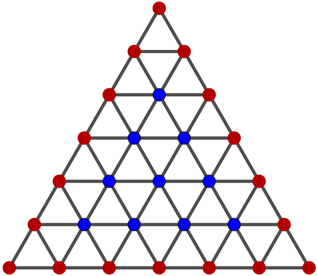
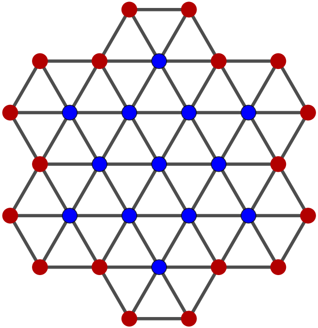
Boundary on graphs!





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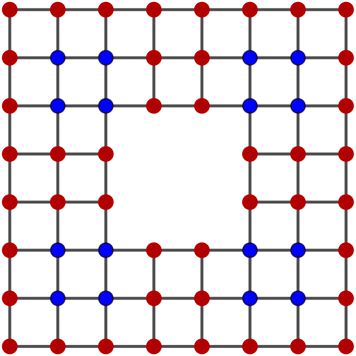

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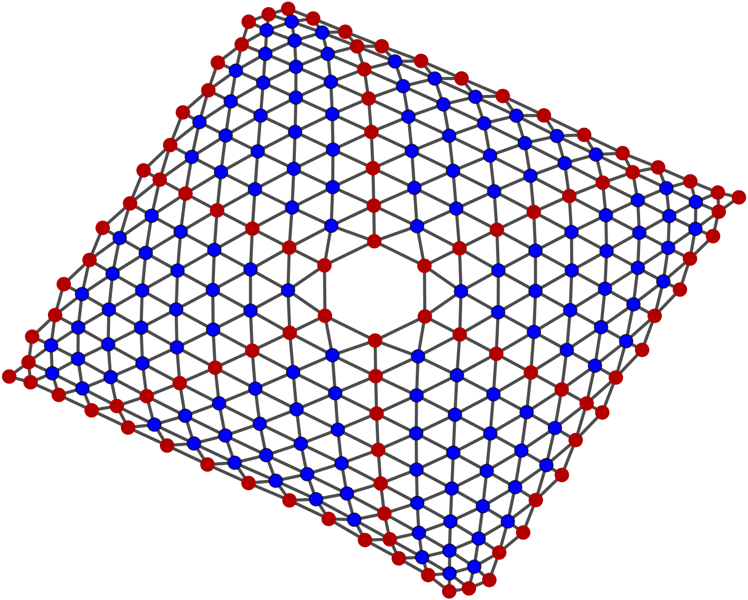

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Among all shapes with fixed area, the disk has the smallest boundary.

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Recall: the diameter is the largest distance between any two points.

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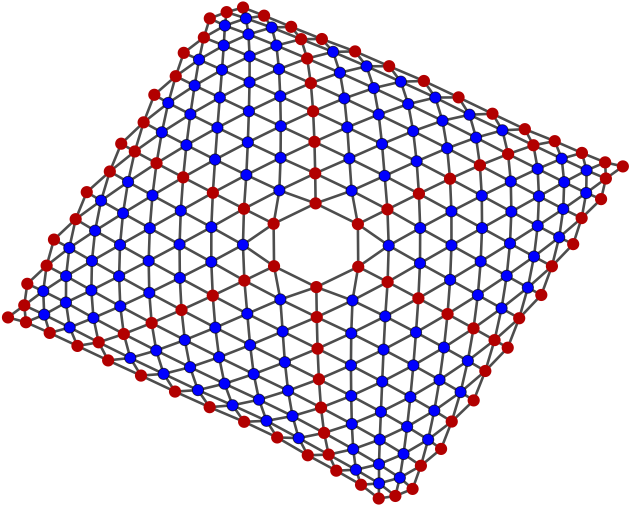
An Isoperimetric Principle for Graphs

If G has maximal degree Δ, then

$$|\partial G| \geq \frac{1}{2\Delta} \cdot \frac{|V|}{\text{diam}(G)}.$$



THANK YOU!