Small Boundaries and Circular Reasoning

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A story so old that even paintings about it are old
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Leonard Limosin, *Dido, Queen of Carthage*, 1564-1565
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*The word limousine is derived from the name of the French region Limousin; however, how the area’s name was transferred to the car is uncertain.* (wikipedia)
A story so old that even paintings about it are old

Guérin, *Dido and Aeneas*, painted 1815
800 BC, political difficulties in Tyre, Dido flees to Tunisia
There she meets King Iarbas and asks him for some land.
Having then bargained for a piece of ground, as much as could be covered with an ox-hide [...]

(Pompeius Trogus, *Historiae Philippicae et Totius Mundi Origines et Terrae Situs*, first Century BC, as excerpted by Marcus Junianus Justinus Frontinus 300 years later)
Having then bargained for a piece of ground, as much as could be covered with an ox-hide [...] she directed the hide to be cut into the thinnest possible strips, and thus acquired a greater portion of ground than she had apparently demanded; [...]

whence the place had afterwards the name of Byrsa.

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(Pompeius Trogus, *Historiae Philippicae et Totius Mundi Origines et Terrae Situs*, first Century BC, as excerpted by Marcus Junianus Justinus Frontinus 300 years later)
At last they landed, where from far your eyes
May view the turrets of new Carthage rise;
There bought a space of ground, which Byrsa call’d,
From the bull’s hide they first inclos’d, and wall’d.

(Vergil, Aeneid, written 29-19 BC)
Byrsa today!
Byrsa today!
Queen Dido’s great insight!

If you have a string of fixed length and want to capture the most area, use a circle.
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If you have a string of fixed length and want to capture the most area, use a circle.

Or: among all sets in the plane with a fixed perimeter, the circle has the largest area.
Civilization VI
Gathering Storm
First Look
Phoenicia
Civ 6 (2019)
I am a Phoenician princess, carrying the name Elissa, who fled my home in Tyre. Having escaped my brother Pygmalion, who murdered my beloved husband Acerbas, a priest of Hercules, I arrived in North Africa.

With my husband’s riches, I bargained with the Berber king Jarnos, who wanted to marry me. He mockingly promised me all the land that I could cover with the skin of a dead ox. I cut the hide into thin pieces and, along with my Tyrian settlers, laid out the borders of my beloved city.

I am ready to become Dido the Wanderer, Queen of Carthage, goddess to my people.
Tunisian Dinar
Fun with language!

Having then bargained for a piece of ground, as much as could be covered with an ox-hide [...] .

(Trogus as excerpted by Justinus)
Fun with language!

Having then bargained for a piece of ground, as much as could be **covered** with an ox-hide [...] .

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Maybe ‘covered’ should have been translated as ‘surrounded’.
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Vergil (Aeneid) writes originally

*Taurino quantum possent circumdare tergo.*
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Maybe ‘covered’ should have been translated as ‘surrounded’. Vergil (Aeneid) writes originally

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Note that

*to surround (circumdare): ‘giving (dare) a circle (circus)’*
Nature has known this for much longer
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(Matthäus Meridan, Map of Paris, 1615)
Cat circles!
Cat circles!
Cat circles!
Cat circles!
Nature loves circles and balls
and hexagons!
The first rigorous understanding

Jakob Steiner (1796-1863)
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization
Steiner symmetrization

$S_L$ Symmetrize about the line $L$

$\Omega$

$S_L(\Omega)$
Steiner symmetrization

Theorem (Jakob Steiner)
This process keeps the area the same but makes the perimeter smaller (or leaves it the same).
Steiner’s argument

'This process keeps the area the same and makes the perimeter smaller (or leaves it the same). The process always gives a new shape unless we have a circle.
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‘This process keeps the area the same and makes the perimeter smaller (or leaves it the same). The process always gives a new shape unless we have a circle.

Therefore the circle is the best shape because it is the only one that does not change.’
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Therefore the circle is the best shape because it is the only one that does not change.’

*Is this a good reason?*
If there is an optimal shape, then it has to be a circle...

**Figure:** Oskar Perron (1880 - 1975)
If there is an optimal shape, then it has to be a circle...

**Perron’s Theorem.** $N = 1$ is the largest integer.

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**Perron’s Theorem.** $N = 1$ is the largest integer.

**Proof.** $N^2 > N$ is always bigger unless $N = 1$.

*Figure: Oskar Perron (1880 - 1975)*
If there is an optimal shape, then it has to be a circle...

**Perron’s Theorem.** $N = 1$ is the largest integer.

**Proof.** $N^2 > N$ is always bigger unless $N = 1$.

...and maybe there is no optimal shape?

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**Figure:** Oskar Perron (1880 - 1975)
Blaschke Selection Theorem
Any sequence of bounded convex domains has a convergent subsequence.

Figure: Wilhelm Blaschke (1885-1962)
Blaschke Selection Theorem
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which concludes the argument.

Figure: Wilhelm Blaschke (1885-1962)
The notion of boundary

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- A point \( x \in \mathbb{R}^n \) is called a **boundary point** of \( S \) if every ball centered at \( x \) contains both points in \( S \) and points in \( S^c \). (Note that if \( x \) is a boundary point of \( S \), \( x \) may belong to either \( S \) or \( S^c \).) The set of all boundary points of \( S \) is called the **boundary** of \( S \) and is denoted by \( \partial S \):

\[
\partial S = \{ x \in \mathbb{R}^n : B(\tau, x) \cap S \neq \emptyset \text{ and } B(\tau, x) \cap S^c \neq \emptyset \text{ for every } \tau > 0 \}.
\]

(Folland, Advanced Calculus)
Some sets are so crazy everything is boundary
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The Mandelbrot set
New challenges: data science!
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BOUNDARY ESTIMATION FROM POINT CLOUDS: ALGORITHMS, GUARANTEES AND APPLICATIONS

JEFF CALDER, SANGMIN PARK, AND DEJAN SLEPČEV
New challenges: data science!

BOUNDARY ESTIMATION FROM POINT CLOUDS: ALGORITHMS, GUARANTEES AND APPLICATIONS

JEFF CALDER, SANGMIN PARK, AND DEJAN SLEPČEV
Can one define the boundary of a graph?

A graph is the collection of points (the *vertices*) some of which are connected (by *edges*) and others are not. Example: people on a social network and whether they are friends or not.
Graphs are thought of abstract ideals
Chartrand, Erwin, Johns, Zhang boundary

There already exists a notion of boundary $(\partial G)^* \subseteq V$ introduced by Chartrand-Erwin-Johns-Zhang introduced in the early 2000s.
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**Figure:** \(u\) is a boundary vertex because \(d(w, u) = 2\) and every neighbor of \(u\) is at most distance 2 from \(w\).
Figure: Graphs, their boundary \((\partial G)^*\) (red) and interior \(V \setminus (\partial G)^*\) (blue).
The CEJZ boundary is pretty natural: if points end up being endpoints in the sense discussed, they should be in the boundary (whatever the definition of boundary ends up being).
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![Diagram](image)

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![Diagram](image)
Let us take a step back and go back to Euclidean space.
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Let us take a step back and go back to Euclidean space.

The boundary is not actually being classified as being part of the boundary since there are always points that are further away.
CEJZ boundary visualized!

One way of visualizing the CEJZ boundary is to fix a vertex $w$ and sort all other vertices by distance from $w$. 
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Figure: Vertices at distance $i-1, i, i+1$ from $w$. 
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A new type of boundary

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Formal definition is (don’t read it)
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Formal definition is (don’t read it)

$$\partial G = \left\{ u \in V \mid \exists w \in V : \frac{1}{\deg(u)} \sum_{(u,v) \in E} d(w, v) < d(u, w) \right\}.$$
Figure: Graphs, their boundary $\partial G$ (red) and interior $V \setminus \partial G$ (blue).

This boundary $\partial G$ always contains the CEJZ boundary $(\partial G)^*$ and sometimes they coincide (as in these pictures).
Figure: Left: Chartrand-Erwin-Johns-Zhang boundary \((\partial G)^*\) (in red). Right: our notion of boundary \(\partial G\) (in red).
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Isoperimetric Principle (Dido’s Principle)
Among all shapes with fixed area, the disk has the smallest boundary.
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Isodiametric Principle
Among all shapes in the plane with fixed area and diameter, the disk has the smallest surface area.
Recall: the diameter is the largest distance between any two points.
Isodiametric Principle

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**Isodiametric Principle**

Among all shapes in the plane with fixed *area* and *diameter*, the disk has the smallest surface area.

If we take a disk of radius $r$, say $D_r$, then

$$ \text{area}(D_r) = r^2 \pi, \quad \text{boundary}(D_r) = 2r \pi $$

and the diameter is $2r$. 
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$$\text{boundary}(D_r) \geq 4 \cdot \frac{\text{area}(D_r)}{\text{diam}(D_r)}.$$
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**Isodiametric Principle**

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If $G$ has maximal degree $\Delta$, then

$$|\partial G| \geq \frac{1}{2}\Delta \cdot |V| \cdot \text{diam}(G).$$
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and in complete analogy

An Isoperimetric Principle for Graphs
If $G$ has maximal degree $\Delta$, then

$$|\partial G| \geq \frac{1}{2\Delta} \cdot \frac{|V|}{\text{diam}(G)}.$$
Thank you!