You're my better half

a tale of complimentary complementary complementary sequences
Preliminaries

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Integer sequences are functions with inputs given by the counting numbers.
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\[(1,2,3,4,5,6,\ldots) = (n)_{n=1}^{\infty}\] (counting numbers)
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\[(2,4,6,8,10,12,\ldots)\]
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Examples.

\[
\begin{align*}
(1,2,3,4,5,6,\ldots) &= (n)_{n=1}^\infty \\
(2,4,6,8,10,12,\ldots) &= (2n)_{n=1}^\infty \\
(1,4,9,16,25,36,\ldots) &= (n^2)_{n=1}^\infty \\
(1,3,6,10,15,21\ldots) &= (n(n + 1)/2)_{n=1}^\infty \\
(1,1,2,3,5,8,13\ldots) &=
\end{align*}
\]

(counting numbers)  
(even numbers)  
(square numbers)  
(triangular numbers)
An integer sequence is an ordered list of integers.

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\[
1, 2, 3, 4, 5, 6, \ldots = (n)_{n=1}^\infty \quad \text{(counting numbers)}
\]

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2, 4, 6, 8, 10, 12, \ldots = (2n)_{n=1}^\infty \quad \text{(even numbers)}
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1, 4, 9, 16, 25, 36, \ldots = (n^2)_{n=1}^\infty \quad \text{(square numbers)}
\]

\[
1, 3, 6, 10, 15, 21, \ldots = (n(n + 1)/2)_{n=1}^\infty \quad \text{(triangular numbers)}
\]

\[
1, 1, 2, 3, 5, 8, 13, \ldots = (F_n)_{n=1}^\infty \quad \text{(Fibonacci numbers)}
\]

\[
F_n = F_{n-1} + F_{n-2}
\]
A real number is **irrational** if it cannot be written as the ratio of two counting numbers.
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Examples.

\[ \sqrt{2} = \frac{a}{b} \]
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\[ \sqrt{2} = \frac{a}{b} \] is impossible if we require \( a \) and \( b \) to be counting numbers.
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Proving a number is irrational is generally hard.

For instance, we don't know if \( e + \pi \) is irrational.
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The floor function: \( \lfloor r \rfloor = \) greatest integer less than or equal to \( r \)
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The floor function: \([ r ] = \text{greatest integer less than or equal to } r\)

\[|\sqrt{2}| = 1 \quad |e| = 2 \quad |\pi| = 3\]
Consider multiples of $\sqrt{2}$
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0, $\sqrt{2}, 2\sqrt{2}$
Consider multiples of $\sqrt{2}$

0, $\sqrt{2}$, $2\sqrt{2}$, $3\sqrt{2}$
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\[
\left(\left\lfloor n \cdot \sqrt{2} \right\rfloor\right)_{n=1}^\infty = (1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, \ldots)
\]
Consider multiples of $\sqrt{2}$

\[
\left(\left\lfloor n \cdot \sqrt{2} \right\rfloor\right)_{n=1}^{\infty} = (1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, ...)
\]

A mystery: what numbers are skipped?
Finding the missing numbers

\[
\left( \left\lfloor n \cdot \sqrt{2} \right\rfloor \right)_{n=1}^{\infty} = (1, 2, 4, 5, 7, 8, 9, 11, 12, 14, 15, 16, 18, 19, ...)
\]

Construct the analogous sequence of multiples of \( \frac{\sqrt{2}}{\sqrt{2} - 1} \)

\[
\left( \left\lfloor n \cdot \frac{\sqrt{2}}{\sqrt{2} - 1} \right\rfloor \right)_{n=1}^{\infty} = (3, 6, 10, 13, 17, 20, ...)
\]
Finding the missing numbers

(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, ...)

Every positive integer appears once and only once!
Did we get lucky with $\sqrt{2}$ ?
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\[
\left(\lfloor n \cdot e \rfloor \right)_{n=1}^{\infty} =
\]
Did we get lucky with $\sqrt{2}$?

$$(\lfloor n \cdot e \rfloor)_{n=1}^{\infty} = (2, 5, 8, 10, 13, 16, 19, 21, \ldots)$$
Did we get lucky with $\sqrt{2}$?

$$\left(\left\lfloor n \cdot e \right\rfloor \right)_{n=1}^{\infty} = (2, 5, 8, 10, 13, 16, 19, 21, ... )$$

$$\left(\left\lfloor n \cdot \frac{e}{e-1} \right\rfloor \right)_{n=1}^{\infty} =$$
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\]

\[
\left\lfloor \frac{n \cdot e}{e - 1} \right\rfloor_{n=1}^{\infty} = (1, 3, 4, 6, 7, 9, 11, 12, 14, 15, 17, 18, 20, 22, 23, ...)
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Every integer appears once and only once!
Can this really happen again?

Try it with your favorite irrational number larger than 1
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\[ x = \pi \text{ and } z = \frac{\pi}{\pi - 1} \]
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In the Sage Cell online:
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\[ x = \pi \text{ and } z = \frac{\pi}{\pi - 1} \quad \quad x = \phi \text{ and } z = \frac{\phi}{\phi - 1} \]

In the Sage Cell online:

```python
x=pi
z=x/(x-1)
print([floor(i*x) for i in [1..20]])
print([floor(i*z) for i in [1..20]])
```
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$$\left(\lfloor n \cdot x \rfloor\right)_{n=1}^{\infty} \text{ and } \left(\lfloor n \cdot z \rfloor\right)_{n=1}^{\infty}$$

divide the counting numbers into two parts with no elements in common.
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**Definition.** Sequences of the form $\left(\lfloor n \cdot x \rfloor\right)_{n=1}^\infty$ are known as *Beatty sequences*. 
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We say the two sequences in Rayleigh's theorem are *complementary*. 
What's going on?

We say the two sequences in Rayleigh's theorem are **complimentary**.
How do we know Rayleigh's theorem is true?

Given the Beatty sequences of interest, \( \left( \lfloor n \cdot x \rfloor \right)_{n=1}^{\infty} \) and \( \left( \lfloor n \cdot \frac{x}{x-1} \rfloor \right)_{n=1}^{\infty} \), we
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\frac{1}{\sqrt{2}} + \frac{1}{\frac{\sqrt{2}}{\sqrt{2}-1}} = \frac{1}{\sqrt{2}} + \frac{\sqrt{2} - 1}{\sqrt{2}} = \frac{1 + \sqrt{2} - 1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1
\]
No Collisions

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\[ y \]
\[ m \cdot \sqrt{2}/(\sqrt{2} - 1) \]
No Collisions

Suppose there exists a counting number $y$ with $\lfloor n \cdot \sqrt{2} \rfloor = y = \lfloor m \cdot \sqrt{2}/(\sqrt{2} - 1) \rfloor$.

Then, we conclude $y < n \cdot \sqrt{2} < y + 1$ and $y < m \cdot \frac{\sqrt{2}}{\sqrt{2} - 1} < y + 1$. 

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Why are the inequalities strict?
No Collisions

Suppose there exists a counting number $y$ with $\lfloor n \cdot \sqrt{2} \rfloor = y = \lfloor m \cdot \frac{\sqrt{2}}{(\sqrt{2} - 1)} \rfloor$

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\[ y \cdot \frac{\sqrt{2}}{(\sqrt{2} - 1)} \]

Then, we conclude $y < n \cdot \sqrt{2} < y + 1$ and $y < m \cdot \frac{\sqrt{2}}{\sqrt{2} - 1} < y + 1$

Why are the inequalities strict?

- $\sqrt{2}$ and $\frac{\sqrt{2}}{(\sqrt{2} - 1)}$ are both irrational;
- any nonzero integer multiple of an irrational is also irrational.
No Collisions

\[ y < n \cdot \sqrt{2} < y + 1 \quad \text{and} \quad y < m \cdot \frac{\sqrt{2}}{\sqrt{2} - 1} < y + 1 \]
No Collisions

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\[ \frac{y}{\sqrt{2}} + \frac{y(\sqrt{2} - 1)}{\sqrt{2}} < n + m < \frac{y + 1}{\sqrt{2}} + \frac{(y + 1)(\sqrt{2} - 1)}{\sqrt{2}} \]
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\( y < n + m < y + 1 \) counting numbers
Suppose there exists a counting number $y$ not in either sequence.
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Then, we conclude

\[ n\sqrt{2} < y \]

\[ y + 1 < (n + 1)\sqrt{2} \]
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Then, we conclude

\[
\begin{align*}
n\sqrt{2} &< y & m\frac{\sqrt{2}}{\sqrt{2} - 1} &< y \\
y + 1 &< (n + 1)\sqrt{2} & y + 1 &< (m + 1)\frac{\sqrt{2}}{\sqrt{2} - 1}
\end{align*}
\]
No Whiffs

\[ n\sqrt{2} < y \quad y + 1 < (n + 1)\sqrt{2} \quad m\frac{\sqrt{2}}{\sqrt{2} - 1} < y \quad y + 1 < (m + 1)\frac{\sqrt{2}}{\sqrt{2} - 1} \]
No Whiffs

\[n\sqrt{2} < y\]
\[y + 1 < (n + 1)\sqrt{2}\]

\[m\frac{\sqrt{2}}{\sqrt{2} - 1} < y\]
\[y + 1 < (m + 1)\frac{\sqrt{2}}{\sqrt{2} - 1}\]

\[n < \frac{y}{\sqrt{2}}\]
\[\frac{y + 1}{\sqrt{2}} < n + 1\]
No Whiffs

\[ n\sqrt{2} < y \quad y + 1 < (n + 1)\sqrt{2} \quad m\frac{\sqrt{2}}{\sqrt{2} - 1} < y \quad y + 1 < (m + 1)\frac{\sqrt{2}}{\sqrt{2} - 1} \]

\[ n < \frac{y}{\sqrt{2}} \quad \frac{y + 1}{\sqrt{2}} < n + 1 \quad m < \frac{y(\sqrt{2} - 1)}{\sqrt{2}} \quad \frac{(y + 1)(\sqrt{2} - 1)}{\sqrt{2}} < m + 1 \]
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\[ n\sqrt{2} < y \quad y + 1 < (n + 1)\sqrt{2} \quad m\frac{\sqrt{2}}{\sqrt{2} - 1} < y \quad y + 1 < (m + 1)\frac{\sqrt{2}}{\sqrt{2} - 1} \]

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(1, 4, 9, 16, 25, 36, 49, ...)
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Question: Are there other methods to generate complementary sequences?

Yes! Let's discuss a general method.
How to find complementary sequences

Suppose that \( f(n) \) is an increasing integer sequence, such as \( f(n) = 2n \) plotted below.
How to find complementary sequences

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How to find complementary sequences

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Build a new sequence \( f^\downarrow \) where \( f^\downarrow(n) \) counts the outputs of \( f \) less than \( n \).

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Suppose that $f(n)$ is an increasing integer sequence, such as $f(n) = 2n$ plotted below.

Build a new sequence $f^{↓}(n)$ where $f^{↓}(n)$ counts the outputs of $f$ less than $n$.

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Suppose that $f(n)$ is an increasing integer sequence, such as $f(n) = 2n$ plotted below.

![Graph showing the sequence $f(n) = 2n$]

Build a new sequence $f\downarrow$ where $f\downarrow(n)$ counts the outputs of $f$ less than $n$.

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Suppose that $f(n)$ is an increasing integer sequence, such as $f(n) = 2n$ plotted below.

![Graph showing increasing sequence $f(n) = 2n$]

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Construct the sequences $f(n) + n$ and $f^\downarrow(n) + n$.

<table>
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<tr>
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<th>1</th>
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How to find complementary sequences

Build a new sequence $f^↓$ where $f^↓(n)$ counts the outputs of $f$ less than $n$.

$$f(n) = 2n$$

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<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>18</td>
<td>20</td>
<td>22</td>
<td>24</td>
</tr>
<tr>
<td>$f^↓(n)$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>4</td>
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Construct the sequences $f(n) + n$ and $f^↓(n) + n$.

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How to find complementary sequences

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Theorem [Lambek/Moser]. Given an increasing integer sequence \( f(n) \), the two integer sequences \( f(n) + n \) and \( f^\downarrow(n) + n \) are complementary.
Theorem [Lambek/Moser]. Given an increasing integer sequence $f(n)$, the two integer sequences $f(n) + n$ and $f^\downarrow(n) + n$ are complementary.

Try it yourself with the increasing sequence $f(n) = n^2$

or

your favorite increasing integer sequence!
Bonus: Beatty Sequences and a Game

Wythoff's game (equivalent version)
Bonus: Beatty Sequences and a Game

Wythoff's game (equivalent version)

Players alternate moving queen on $m \times n$ board:
Bonus: Beatty Sequences and a Game

Wythoff's game (equivalent version)

Players alternate moving queen on $m \times n$ board:

Valid moves:
Bonus: Beatty Sequences and a Game

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Players alternate moving queen on $m \times n$ board:

Valid moves:

1. Any number of spaces right
Bonus: Beatty Sequences and a Game

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Players alternate moving queen on $m \times n$ board:

Valid moves:

1. Any number of spaces right
2. Any number of spaces down
Bonus: Beatty Sequences and a Game

Wythoff's game (equivalent version)

Players alternate moving queen on $m \times n$ board:

Valid moves:

1. Any number of spaces right
2. Any number of spaces down
3. Any number of spaces SE diagonal
Bonus: Beatty Sequences and a Game

Wythoff's game (equivalent version)

Players alternate moving queen on $m \times n$ board:

Valid moves:

1. Any number of spaces right
2. Any number of spaces down
3. Any number of spaces SE diagonal

Player who moves queen to bottom right square wins!
Bonus: Beatty Sequences and a Game

Wythoff's game strategy:
Wythoff's game strategy:

\[ \left( \left\lfloor n \cdot \phi \right\rfloor \right)_{n=0}^{\infty} = (0, 1, 3, 4, 6, 8, 9, ...) \]

\[ \left( \left\lfloor n \cdot \phi/(\phi - 1) \right\rfloor \right)_{n=0}^{\infty} = (0, 2, 5, 7, 10, 13, 15, ...) \]
Wythoff's game strategy:

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\begin{align*}
\left(\left\lfloor n \cdot \phi \right\rfloor\right)_{n=0}^\infty &= (0, 1, 3, 4, 6, 8, 9, \ldots) \\
\left(\left\lfloor n \cdot \phi / (\phi - 1) \right\rfloor\right)_{n=0}^\infty &= (0, 2, 5, 7, 10, 13, 15, \ldots)
\end{align*}
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**Bonus: Beatty Sequences and a Game**

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\left( \lfloor n \cdot \phi \rfloor \right)_{n=0}^\infty = (0, 1, 3, 4, 6, 8, 9, \ldots) \quad \text{Strategy: move to red square}
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Bonus: Beatty Sequences and a Game

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Bonus: Rational Beatty Sequences

\[
\left(\left\lfloor n \cdot \frac{3}{2} \right\rfloor\right)_{n=1}^{\infty} = (1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, \ldots)
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Bonus: Rational Beatty Sequences

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\]

Differences:
Bonus: Rational Beatty Sequences

\[
\left(\left\lfloor n \cdot \frac{3}{2} \right\rfloor\right)_{\infty}^{n=1} = (1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, \ldots)
\]

Differences: \(2, \)
Bonus: Rational Beatty Sequences

$$\left( \left\lfloor n \cdot \frac{3}{2} \right\rfloor \right)_{n=1}^\infty = (1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, \ldots)$$

Differences: (2, 1,
**Bonus: Rational Beatty Sequences**

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Differences: \((2, 1, 2,\ldots)\)
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\]

Differences: \( (2, 1, 2, 1, 2, \ldots) \)
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Differences: \( (2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots) \)
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Differences: \((2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots)\)

Repeats 2,1 forever
Bonus: Rational Beatty Sequences

\[
\left(\left\lfloor n \cdot \frac{3}{2}\right\rfloor\right)_{n=1}^\infty = (1, 3, 4, 6, 7, 9, 10, 12, 13, 15, 16, 18, 19, 21, 22, 24, 25, 27, 28, ...)
\]

Differences: \((2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)\)

Repeats 2,1 forever

**Fact:** \(x\) is rational if and only if the first difference sequence of \(\left(\left\lfloor nx\right\rfloor\right)_{n=1}^\infty\) is periodic.
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

\((2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots)\)
Bonus: Rational Beatty Sequences

\[(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots)\]
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

$(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)\)
Bonus: Rational Beatty Sequences

\[2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots\]
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Bonus: Rational Beatty Sequences

\((2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots)\)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Diagram of the sequence:

0
|-- 3
|   |-- 6
|       |-- 9
|           |-- 15
|               |-- 24
|                   |-- 36
|       |-- 12
|           |-- 18
|               |-- 27
|                   |-- 39
|                       |-- 42
|                           |-- 45
|                               |-- 48
|       |-- 21
|           |-- 33

Note: The diagram visualizes the sequence and its structure, not the textual description.
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Diagram of a tree with nodes labeled with numbers, illustrating the sequence.
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Diagram:

- Node 0
  - Node 2
    - Node 3
      - Node 6
        - Node 9
          - Node 12
            - Node 15
              - Node 18
                - Node 21
                  - Node 24
                    - Node 27
                      - Node 30
                        - Node 33
                          - Node 36
                            - Node 39
                              - Node 42
                                - Node 45
                                  - Node 48
                                    - Node 51
                                      ...
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Diagram of the rational Beatty sequence.
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Diagram showing the construction of the sequence.
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

Diagram showing a tree structure with nodes labeled with numbers, illustrating the sequence: 0, 2, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, ...
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

```
(0, 2, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, ...
```
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

...
**Bonus: Rational Beatty Sequences**

\[(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, \ldots)\]

\[42 = 2 \cdot \left(\frac{3}{2}\right)^6 + 1 \cdot \left(\frac{3}{2}\right)^5 + 2 \cdot \left(\frac{3}{2}\right)^4 + 0 \cdot \left(\frac{3}{2}\right)^3 + 0 \cdot \left(\frac{3}{2}\right)^2 + 1 \cdot \left(\frac{3}{2}\right)^1 + 0 \cdot \left(\frac{3}{2}\right)^0\]
Bonus: Rational Beatty Sequences

\((2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...\)

\[42 = 2 \cdot \left( \frac{3}{2} \right)^6 + 1 \cdot \left( \frac{3}{2} \right)^5 + 2 \cdot \left( \frac{3}{2} \right)^4 + 0 \cdot \left( \frac{3}{2} \right)^3 + 0 \cdot \left( \frac{3}{2} \right)^2 + 1 \cdot \left( \frac{3}{2} \right)^1 + 0 \cdot \left( \frac{3}{2} \right)^0\]

\[43 = 2 \cdot \left( \frac{3}{2} \right)^6 + 1 \cdot \left( \frac{3}{2} \right)^5 + 2 \cdot \left( \frac{3}{2} \right)^4 + 0 \cdot \left( \frac{3}{2} \right)^3 + 0 \cdot \left( \frac{3}{2} \right)^2 + 1 \cdot \left( \frac{3}{2} \right)^1 + 1 \cdot \left( \frac{3}{2} \right)^0\]
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)
Bonus: Rational Beatty Sequences

(2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 1, ...)

This tree encodes the "base-3/2 representations" of counting numbers - each number is given as a sum of powers of 3/2, with coefficients 0, 1, or 2.

\[
42 = 2 \cdot \left( \frac{3}{2} \right)^{6} + 1 \cdot \left( \frac{3}{2} \right)^{5} + 2 \cdot \left( \frac{3}{2} \right)^{4} + 0 \cdot \left( \frac{3}{2} \right)^{3} + 0 \cdot \left( \frac{3}{2} \right)^{2} + 1 \cdot \left( \frac{3}{2} \right) + 0 \cdot \left( \frac{3}{2} \right)^{0}
\]

\[
43 = 2 \cdot \left( \frac{3}{2} \right)^{6} + 1 \cdot \left( \frac{3}{2} \right)^{5} + 2 \cdot \left( \frac{3}{2} \right)^{4} + 0 \cdot \left( \frac{3}{2} \right)^{3} + 0 \cdot \left( \frac{3}{2} \right)^{2} + 1 \cdot \left( \frac{3}{2} \right) + 1 \cdot \left( \frac{3}{2} \right)^{0}
\]

\[
44 = 2 \cdot \left( \frac{3}{2} \right)^{6} + 1 \cdot \left( \frac{3}{2} \right)^{5} + 2 \cdot \left( \frac{3}{2} \right)^{4} + 0 \cdot \left( \frac{3}{2} \right)^{3} + 0 \cdot \left( \frac{3}{2} \right)^{2} + 1 \cdot \left( \frac{3}{2} \right) + 2 \cdot \left( \frac{3}{2} \right)^{0}
\]
Thanks for listening!
If you have questions, I am happy to answer them.

Tom Edgar : edgartj@plu.edu