Mildly Impressive Mathematical Card Tricks

Dr. Jonah Ostroff

Monthly Math Hour

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The Trick

- Audience member gives me five cards.
- I lay 4 of them out on a table.
- Kolya guesses the fifth.

How Is This Possible?

Number of remaining cards: 48

Number of permutations: $4! = 24$

$(4 \times 3 \times 2 \times 1)$

The Five-Card Trick Solution

Idea: In any 5-card hand.... two cards are the same suit!

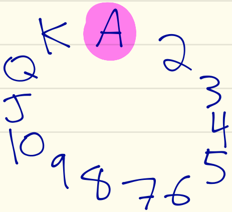
How far apart can two cards be?

12?

A 2 3 4 5 6 7 8 9 10 J Q K

12 apart

or:



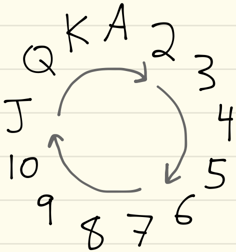
6 apart!

The Five-Card Trick Solution

(William Fitch Cheney, 1950)

The Arranger:

- Two of the cards share a suit, right?
- Awesome. Hide the one that is 1-6 higher than the other, using the cyclic ordering.
- Put the "lower" card first. Arrange the others so to convey how much higher the hidden card is.



The Guesser:

- Look at the order of the second, third, and fourth cards to get a number from 1 to 6.
- Add that to the first card (wrapping K→A→2 if necessary) to get the answer, in the same suit.

Tiebreaker!

♣ ♦ ♥ ♠

C < D < H < S

5 Clubs
4 Spades
7 Hearts
Q Hearts
3 Diamonds

7H 5C 3D 4S

Six orderings:

L = lowest card
M = middle card
H = highest card

□ L M H : 1
□ L H M : 2
□ M L H : 3
□ M H L : 4
□ H L M : 5
□ H M L : 6

The Four-Card Trick Solution

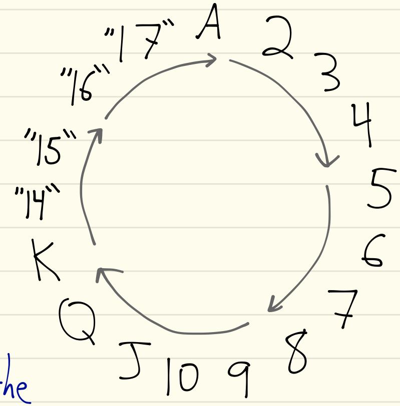
Too many suits! Let's split up the spades, merging them with the other suits:

17 "Clubs"	17 "Diamonds"	17 "Hearts"
A ♣	A ♦	A ♥
2 ♣	2 ♦	2 ♥
3 ♣	3 ♦	3 ♥
4 ♣	4 ♦	4 ♥
5 ♣	5 ♦	5 ♥
6 ♣	6 ♦	6 ♥
7 ♣	7 ♦	7 ♥
8 ♣	8 ♦	8 ♥
9 ♣	9 ♦	9 ♥
10 ♣	10 ♦	10 ♥
J ♣	J ♦	J ♥
Q ♣	Q ♦	Q ♥
K ♣	K ♦	K ♥
2 ♠	6 ♠	10 ♠
3 ♠	7 ♠	J ♠
4 ♠	8 ♠	Q ♠
5 ♠	9 ♠	K ♠

The Four-Card Trick Solution

The Arranger:

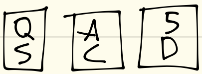
- Pick two cards from the same "suit."
- Hide the one that's 1-8 ranks higher than the other, cyclically.
- Arrange the three cards to indicate the difference, as shown below. Put the base card in position ☆.

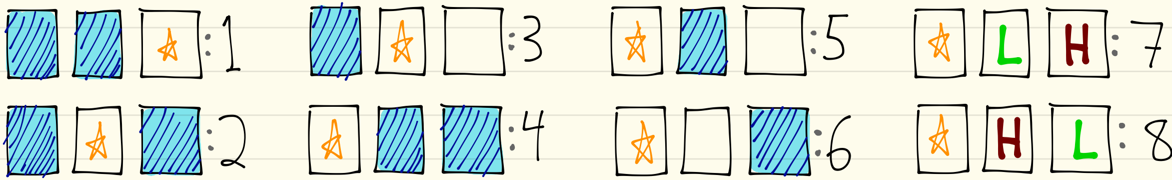


The Guesser:

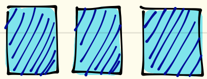
- Interpret the arrangement as a number from 1 to 8, as shown below.
- Add that number to the base card ☆ in the same "suit", wrapping around from "17" to Ace.

A Clubs
5 Diamonds
6 Hearts
Q Spades





What about the Ace of Spades?



How Many Cards Can We Have in the Deck?

Say we have n cards

Number of 5-card hands $\leq$ Number of 4-card messages

$$\frac{n(n-1)(n-2)(n-3)(n-4)}{120} \qquad n(n-1)(n-2)(n-3)$$

order doesn't matter

$$\frac{\cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}(n-4)}{120} \leq \cancel{n}\cancel{(n-1)}\cancel{(n-2)}\cancel{(n-3)}$$

$$n-4 \leq 120$$

$$n \leq 124$$

Is the 124-Card Deck Trick Possible?

Need a matching between hands and messages.

Not every matching works!

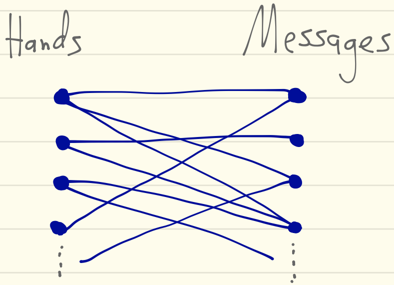
Hand: {1, 37, 46, 90, 112}

↕

Message: 90 37 1 68

no 68

We need a matching on a bipartite graph:

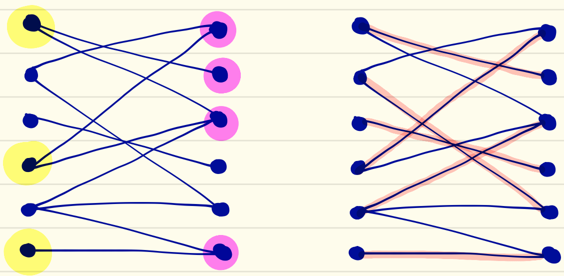


Hall's Matching Theorem

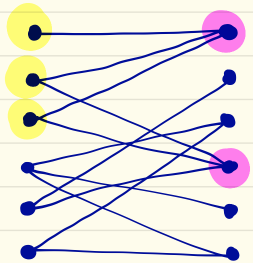
(Philip Hall, 1935)

A bipartite graph has a perfect matching if and only if every set of n points has at least n neighbors.

Ex:



Ex:



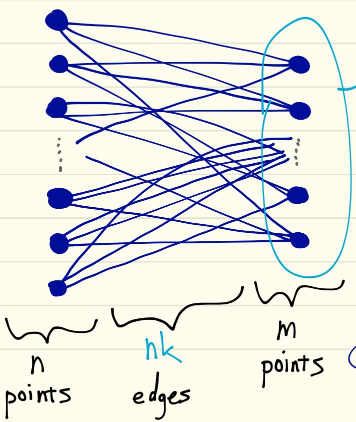
no matching!

Corollary to Hall's Matching Theorem

If every point in a bipartite graph has the same number of neighbors (k, say), then the graph has a perfect matching.

<u>Proof:</u> We'll show that if every point has k neighbors then every set of n points has at least n neighbors. (So Hall's matching theorem applies and we're done.)

By contradiction, suppose some group of n points has m neighbors, and $m < n$:



→ have at least $\frac{nk}{m}$ edges per point on average $> k$

Contradiction!

Back to cards:

- How many messages could you make from one 5-card hand? $5 \times 4 \times 3 \times 2 = 120$
- How many 5-card hands could form each message? 120

33 36 101 5

53

A Modular Idea

"mod 5" arithmetic:

$3+4 \equiv 2 \pmod 5$

$2+3 \equiv 0 \pmod 5$

$6+1 \equiv 2 \pmod 5$

$18+26 \equiv 4 \pmod 5$

$107+55 \equiv 2 \pmod 5$

- Look at the sum of the 5 numbers, mod 5.
- If it's:
  - 0 → hide the lowest number
  - 1 → hide the second-lowest number
  - 2 → hide the third-lowest number
  - 3 → hide the fourth-lowest number
  - 4 → hide the largest number

Result: Suppose you're the guesser and you see these four numbers: sum is 4 (mod 5)

13 35 64 97

Possible answers: 1, 6, 11 | 17, 22, 27, 32 | 38, 43, ..., 63 | 69, 74, ..., 94 | 100, 105, ..., 120

24 total

24 permutations

1 2 3 4 : 1	2 1 3 4 : 7	3 1 2 4 : 13	4 1 2 3 : 19
1 2 4 3 : 2	2 1 4 3 : 8	3 1 4 2 : 14	4 1 3 2 : 20
1 3 2 4 : 3	2 3 1 4 : 9	3 2 1 4 : 15	4 2 1 3 : 21
1 3 4 2 : 4	2 3 4 1 : 10	3 2 4 1 : 16	4 2 3 1 : 22
1 4 2 3 : 5	2 4 1 3 : 11	3 4 1 2 : 17	4 3 1 2 : 23
1 4 3 2 : 6	2 4 3 1 : 12	3 4 2 1 : 18	4 3 2 1 : 24

Five-Number Trick Solution

The Arranger:

- Compute the mod 5 sum of the numbers.
- If it's 0, 1, 2, 3, or 4, then hide the lowest, second-lowest, ..., or greatest number.
- Take the hidden number, and subtract one for each non-hidden number that's smaller than it.
- Divide by 5, rounding up. You'll get a result between 1 and 24.
- Arrange the 4 non-hidden numbers to express that result.

The Guesser:

- Interpret the permutation as a number from 1 to 24.
- Multiply by 5.
- Subtract the mod 5 sum of the four numbers.
- Add 1 for each given number less than your result.