Mildly Impressive Mathematical Card Tricks
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Monthly Math Hour
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The Trick

- Audience member gives me five cards.
- I lay 4 of them out on a table.
- Kolya guesses the fifth.
How Is This Possible?

Number of remaining cards: 48

Number of permutations: $4! = 24$

$(4 \times 3 \times 2 \times 1)$
The Five-Card Trick Solution

Idea: In any 5-card hand,... two cards are the same suit!

How far apart can two cards be?

12?

A 2 3 4 5 6 7 8 9 10 J Q K

12 apart

or: Q K A 2

6 apart!
The Five-Card Trick Solution

(William Fitch Cheney, 1950)

The Arranger:

• Two of the cards share a suit, right?

• Awesome. Hide the one that is 1-6 higher than the other, using the cyclic ordering.

• Put the "lower" card first. Arrange the others so to convey how much higher the hidden card is.

The Guesser:

• Look at the order of the second, third, and fourth cards to get a number from 1 to 6.

• Add that to the first card (wrapping K→A→2 if necessary) to get the answer, in the same suit.

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5 Clubs 4 Spades
7 Hearts Q Hearts
3 Diamonds

6H 5C 3D 4S

Six orderings: L M H:1 M H L:4
L = lowest card
M = middle card
H = highest card

L H M:2 H L M:5
M L H:3 H M L:6
The Four-Card Trick Solution

Too many suits! Let's split up the spades, merging them with the other suits:

17 "Clubs"

17 "Diamonds"

17 "Hearts"
The Four-Card Trick Solution

The Arranger:

- Pick two cards from the same "suit".
- Hide the one that's 1-8 ranks higher than the other, cyclically.
- Arrange the three cards to indicate the difference, as shown below. Put the base card in position $\heartsuit$.

The Guesser:

- Interpret the arrangement as a number from 1 to 8, as shown below.
- Add that number to the base card in the same "suit", wrapping around from "17" to Ace.

What about the Ace of Spades?
How Many Cards Can We Have in the Deck?

Say we have \( n \) cards

Number of 5-card hands \( \leq \) Number of 4-card messages

\[
\frac{n(n-1)(n-2)(n-3)(n-4)}{120} \leq \frac{n(n-1)(n-2)(n-3)}{120}
\]

Order doesn't matter

\[
\frac{n(n-1)(n-2)(n-3)(n-4)}{120} \leq \frac{n(n-1)(n-2)(n-3)}{120}
\]

\( n^4 \leq 120 \)

\( n \leq 124 \)
Is the 124-Card Deck Trick Possible?

Need a matching between hands and messages.

Not every matching works!

Hand: \{ 1, 37, 46, 90, 112 \}

Message: 90 37 1 68

We need a matching on a bipartite graph:

Hands  Messages

\[ \text{Diagram showing a bipartite graph with connections between hands and messages.} \]
A bipartite graph has a perfect matching if and only if every set of $n$ points has at least $n$ neighbors.

Examples:

1. Perfect matching.
2. Perfect matching.
3. No matching.
Corollary to Hall's Matching Theorem

If every point in a bipartite graph has the same number of neighbors, then the graph has a perfect matching.

Proof: We'll show that if every point has \( k \) neighbors then every set of \( n \) points has at least \( n \) neighbors. (So Hall's matching theorem applies, and we're done.)

By contradiction, suppose some group of \( n \) points has \( m \) neighbors and \( m < n \):

\[
\frac{nk}{m} \geq k
\]

Back to cards:

- How many messages could you make from one 5-card hand? \( 5 \times 4 \times 3 \times 2 = 120 \)
- How many 5-card hands could form each message? 120
A Modular Idea

"mod 5" arithmetic:

- Look at the sum of the 5 numbers, mod 5.
- If it's: 0 → hide the lowest number
  - 1 → hide the second-lowest number
  - 2 → hide the third-lowest number
  - 3 → hide the fourth-lowest number
  - 4 → hide the largest number

Result: Suppose you're the guesser and you see these four numbers: sum is 4 (mod 5)

Possible answers: 1, 6, 11, 17, 22, 27, 32, 38, 43, ..., 63, 69, 74, ..., 94, 100, 105, ..., 120

24 total
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24 permutations
The Arranger:

- Compute the mod 5 sum of the numbers.
- If it's 0, 2, 3, or 4, then hide the lowest, second-lowest..., or greatest number.
- Take the hidden number, and subtract one for each non-hidden number that's smaller than it.
- Divide by 5 rounding up. You'll get a result between 1 and 24.
- Arrange the 4 non-hidden numbers to express that result.

The Guesser:

- Interpret the permutation as a number from 1 to 24.
- Multiply by 5.
- Subtract the mod 5 sum of the four numbers.
- Add 1 for each given number less than your result.