A Tale of Knots & Games

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What is a knot?
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(We’ll come back to this.)
Ancient knots in art

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Ancient knots in art

While celtic knots began to appear in history around 450 AD...

...knots have been appearing in art since at least 2200 BC.

Figure: A seal-impression from the House of the Tiles in Lerna.
The story of the Gordian knot

A knot that was impossibly difficult to untie was tied to an oxcart belonging to Gordias.

In 330 BC, Alexander the Great famously tried to untie the knot. Upon failing to solve the puzzle the “correct” way, he unsheathed his sword and sliced the knot in half!
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The Inca empire in fourteenth century South America used knots (quipu) for accounting.
Out to sea!

Knots have been put to use for fishing and sailing for as long as we can remember.
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Knots and chemistry (cont.)

Pb?  Na?  Fe?
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2. Chemistry (properties of molecules)
3. Biology (DNA replication)
What is a knot?

Definition (A Mathematical Notion)

A knot is a circle that doesn’t intersect itself sitting in space.
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A **knot** is a circle that doesn’t intersect itself sitting in space.

Intuitively, we say that two knots are **equivalent** if we can get from one to the other by bending, stretching, and rotating *as long as we don’t break or cut* the knot anywhere.
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Definition (A Mathematical Notion)

A knot is a circle that doesn’t intersect itself sitting in space.

Intuitively, we say that two knots are equivalent if we can get from one to the other by bending, stretching, and rotating as long as we don’t break or cut the knot anywhere. (Sorry, Alexander. No swords allowed!)
What are links?

Definition (A Mathematical Notion)

A **link** is a collection of non-intersecting knots (perhaps linked with one another) sitting in space.
A trivial knot is called the **unknot**.
A trivial knot is called the \textit{unknot}.

A trivial link is called the \textit{unlink}.
Because we like to represent knots using their pictures, we usually equate knots with their knot diagrams.
Knot diagrams

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A knot diagram is a closed curve in the plane containing crossings (no tangencies or triple-points!). We decorate these crossings in a particular way to indicate which is the over-strand and which is the under-strand of the crossing.
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from our viewpoint ...
Knot diagram equivalence

- Problem: There are many different pictures of the same knot.
- For example, we can look at this knot...

from our viewpoint ...

...or from a “bird’s eye” viewpoint.
This knot...
This knot...

...is the unknot in disguise!
How can we show that two diagrams represent the same knot?
In 1927, Kurt Reidemeister showed that knot diagrams are equivalent precisely when they can be related by the following moves.
I’ll make an example of you!

Let’s use the example we looked at before to show how Reidemeister moves work.

![Diagram of knots with arrows showing Reidemeister moves]

This knot, called the figure eight knot, is equivalent to its mirror image. (Prove it!)
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Let’s use the example we looked at before to show how Reidemeister moves work.

![Diagram of knot transformations](image)

At the end of our sequence of moves, we have the mirror image of the diagram we wanted. This knot, called the figure eight knot, is equivalent to its mirror image. (Prove it!)
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At the end of our sequence of moves, we have the mirror image of the diagram we wanted. This knot, called the figure eight knot, is equivalent to its mirror image. (Prove it!)
Just as the same knot can look very different in two different diagrams, different knots can look very similar to one another.
A slippery slope

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Unknotted

Knotted
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Unknotted?  Knotted?
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Unknotted??

Knotted??
Let’s use this idea to play a game!
Starting with a knot that is missing its crossing information, we can play the **Knotting–Unknotting Game**.
Knotting vs. Unknotting

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- In this game, there are two players, **Knot** and **Unknot**.
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- Players take turns choosing crossing information.
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**Knot** wants to make something that is knotted up, while **Unknot** wants to make something that can be untangled.
Playing the game
Knot’s Move

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Unknot’s Move

Who wins?

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Now it’s your turn

1. Get a worksheet and find a partner.

2. Assign roles. (One of you is Knot and one of you is Unknot.)

3. Decide who plays first, then play your first game.

4. Play again on the same “game board,” switching who goes first but keeping the same roles.

5. When you are done, draw your own game board and play another game!

6. Did you learn any strategies?

7. Any observations about which player has an advantage?

8. Did any questions arise?
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Did you learn any strategies? Any observations about which player has an advantage? Did any questions arise?

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Some observations

- If Knot can make the knot *alternating*, she can win.
Some observations

- If *Knot* can make the knot *alternating*, she can win.

- If *Unknot* can make long strands go entirely under or entirely over, the knot can be simplified.
If both players play optimally on this game board, whoever goes first loses. This is true regardless of whether Knot goes first or Unknot goes first!
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What about your game board? Who has a winning strategy?
What other games could we play?
If you start with a knot or a link that is missing its crossing information, you can play the **Link Smoothing Game**.
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- In this game, there are two players: *Knot* and *Link*.
Knotting vs. Linking

If you start with a knot or a link that is missing its crossing information, you can play the **Link Smoothing Game**.

- In this game, there are two players: *Knot* and *Link*.
- Players take turns to select a crossing and **smooth** it:

![Diagram of crossing smoothing](image)
If you start with a knot or a link that is missing its crossing information, you can play the **Link Smoothing Game**.

- In this game, there are two players: *Knot* and *Link*.
- Players take turns to select a crossing and **smooth** it:

  - *Link* wants to disconnect the diagram to get an unlink, while *Knot* wants to keep it all in one piece to get an unknot.
Link’s Move

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Link’s Move

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Who wins?
Now it’s your turn

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Now it’s your turn

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2. Decide who plays first, and play the game.
Now it’s your turn

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3. Play again, switching who goes first.
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Any observations about which player has an advantage?

Did any questions arise?

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Some observations

- *Link* wins if he can play on a **nugatory** crossing.
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- When *Link* plays last, *Link* wins.
Some observations

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- *Link* wins if the diagram contains a picture like this:

- When *Link* plays last, *Link* wins.
- Does *Link* always have the upper hand??
This is an example of a link shadow where *Knot* actually has a winning strategy if she plays second.
A winning strategy

- This is an example of a link shadow where Knot actually has a winning strategy if she plays second.

- More often than not, Link has a winning strategy...
A winning strategy

- This is an example of a link shadow where *Knot* actually has a winning strategy if she plays second.

- More often than not, *Link* has a winning strategy...but we have found infinite families of diagrams on which *Knot* has an advantage.
Keep playing these games and see if you can figure out who has a winning strategy for specific shadows!
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These are just a couple of games you could play using knots and links. Invent your own games!
Unsolicited advice

- **Keep playing these games** and see if you can figure out who has a winning strategy for specific shadows!
- These are just a couple of games you could play using knots and links. **Invent your own games**!
- Are you interested in knowing more about knots? **Read Why Knot?**, a comic book about knots by Colin Adams.

![Image of Why Knot? comic book cover]
thank you.
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