Computer Proofs in Algebra, Combinatorics and Geometry

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May 15, 2011
Outline

1. Example of human proof.
2. Example of computer proof.
3. History of first major computer proof.
4. Recent results
Definition of “Proof”
Definition of “Proof”

• Proof: “An argument or evidence establishing the truth of a statement.”

• From Bing:
  • Definitions of proof (n)
  • proof [ proof ]
  • conclusive evidence: evidence or an argument that serves to establish a fact or the truth of something
  • test of something: a test or trial of something to establish whether it is true
  • state of having been proved: the quality or condition of having been proved
  • Synonyms: resistant, resilient, impervious, immune
Statement

There exists an infinite number of prime numbers.

True or False?
Human Proof

Statement: There exists an infinite number of prime numbers.

Proof: Assume there exists only $N$ distinct primes:

$$1 < p_1 = 2 < p_2 = 3 < p_3 < ... < p_N.$$ 

Set $K = p_1 \cdot p_2 \cdot p_3 \cdot ... \cdot p_N + 1$.

$K$ has a prime factorization say $K = q_1 \cdot q_2 \cdot q_3 \cdot ... \cdot q_M$. 
Proof: Assume there exists only N distinct primes:
1 < p_1 = 2 < p_2 = 3 < p_3 < ... < p_N.
Set K = p_1 \ p_2 \ p_3 \ ... \ p_N + 1.
K has a prime factorization say K = q_1 \ q_2 \ q_3 \ ... \ q_M.

Subtracting
\[
(*) \quad q_1 \ q_2 \ q_3 \ ... q_M - p_1 \ p_2 \ p_3 \ ... \ p_N = 1.
\]

If q_2 = p_j for some j then the left side of (*) is divisible by p_j with no remainder. But the right side of (*) is 1 so it is not divisible by p_j > 1 with no remainder. Contradiction!
Proof: Assume there exists only N distinct primes:
1<\( p_1 = 2 < p_2 = 3 < p_3 < \ldots < p_N \).
Set \( K = p_1 \ p_2 \ p_3 \ldots \ p_N + 1 \).
K has a prime factorization say \( K = q_1 \ q_2 \ q_3 \ldots q_M \).

Subtracting
\[
(*) \quad q_1 \ q_2 \ q_3 \ldots q_M - p_1 \ p_2 \ p_3 \ldots p_N = 1.
\]

If \( q_1 = p_j \) for some \( j \) then the left side of (*) is divisible by \( p_j \) with no remainder. But the right side of (*) is 1 so it is not divisible by \( p_j >1 \) with no remainder. Contradiction!
Theorem: There exists an infinite number of prime numbers.

Proof: Assume there exists only N distinct primes:
\[ 1 < p_1 = 2 < p_2 = 3 < p_3 < ... < p_N. \]
Set \( K = p_1 \cdot p_2 \cdot p_3 \cdots p_N + 1. \)
K has a prime factorization say \( K = q_1 \cdot q_2 \cdot q_3 \cdots q_M \)
Subtracting
\[ (*) \quad q_1 \cdot q_2 \cdot q_3 \cdots q_M - p_1 \cdot p_2 \cdot p_3 \cdots p_N = 1. \]
If \( q_1 = p_j \) for some \( j \) then the left side of (*) is divisible by \( p_j \) with no remainder. But the right side of (*) is 1 so it is not divisible by \( p_j > 1 \) with no remainder. Contradiction!

Q.E.D.
Example of Computer Proof

Use symbolic algebra package like Maple or Mathematica.
Directions: Color the map below with as few colors as possible without choosing the same color for two adjacent states. Use letters like G for green or Y for yellow if you don't have colored pencils.
A 4-coloring of the states

Experiment

Try constructing a map for yourself which requires 5 colors.
Statement

• “Every map of states/countries/counties etc can be colored using 4 colors such that no two adjacent states are given the same color.”

• True or False?

• Caveats: No two states touch at isolated points. Each state is connected.
Statement

“Every map of states/countries/counties etc can be colored using 4 colors such that no two adjacent states are given the same color. “

History:

• 1852: Conjectured to be true by Francis Guthrie (cartographer or botanist).

• Francis Guthrie -> Fredrick Guthrie -> Augustus De Morgan -> Arthur Cayley
History

• 1852: Conjectured to be true by Francis Guthrie
• 1878: Cayley published Guthrie’s conjecture.
• 1879: Kempe published a proof.
• 1880: Tait published a proof.
• 1890: Heawood pointed out a flaw with Kempe’s proof!
• 1891: Petersen pointed out a flaw with Tait’s proof!

... 

• Many proofs and disproofs appear and get rejected. But much progress was made along the way.
History

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• 1878: Cayley published Guthrie’s conjecture.
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• 1890: Heawood pointed out a flaw with Kempe’s proof!
• 1891: Petersen pointed out a flaw with Tait’s proof!
• …. Many proofs and disproofs appear and get rejected. But much progress was made along the way. The field of graph theory was born into mathematics.
• 1976: Appel and Haken publish a highly controversial computer assisted proof. NY Times refuses to mention it.
Reformulation

• Instead of coloring maps, the problem was generalized to coloring planar graphs.

• Replace each state with a bold dot = vertex.
• Connect the dots representing two states if and only if they are adjacent on the map by path on the paper = edge.
Place a bold dot in each state. Each dot is called a **vertex** of the graph.

*Directions:* Color the map below with as few colors as possible without choosing the same color for two adjacent states. Use letters like G for green or Y for yellow if you don't have colored pencils.
Draw a path between every pair vertices representing adjacent states which Only passes through those two states. Each path is called an edge of the graph.

Directions: Color the map below with as few colors as possible without choosing the same color for two adjacent states. Use letters like G for green or Y for yellow if you don't have colored pencils.
Draw a path between every pair vertices representing adjacent states which only passes through those two states. Each path is called an edge of the graph.

Directions: Color the map below with as few colors as possible without choosing the same color for two adjacent states. Use letters like G for green or Y for yellow if you don't have colored pencils.
A graph $G=(V,E)$ is a set of vertices $V$ and a subset of all pairs of vertices $E$.

$G$ is planar if all the edges can be drawn in the plane without crossing each other.

Delete the map. What remains is a planar graph.

Question: Can every map of states be represented by a planar graph?

Question: Is every graph planar?
Four Color Theorem

“Every vertex in a planar graph can be assigned a color distinct from all of its neighbors using at most 4 colors.”
Controversy over Computer Proof

• Imagine back to 1976... ..
PDP-8
Computer built around 1970
This complete PDP-8 assembly language program outputs "Hello, world!" to the teleprinter.

*10                   / Set current assembly origin to address 10,
STPTR,    STRNG-1     / An auto-increment register (one of eight at 10-17)

*200                  / Set current assembly origin to program text area
HELLO,  CLA CLL       / Clear AC and Link again (needed when we loop back from tls)
TAD I Z STPTR / Get next character, indirect via PRE-auto-increment address from the zero page
SNA       / Skip if non-zero (not end of string)
HLT       / Else halt on zero (end of string)
TLS       / Output the character in the AC to the teleprinter
TSF       / Skip if teleprinter ready for character
JMP .-1   / Else jump back and try again
JMP HELLO / Jump back for the next character
STRNG,  310           / H
  345     / e
  354     / l
  354     / l
  357     / o
  254     / ,
  240     / (space)
  367     / w
  357     / o
  362     / r
  354     / l
  344     / d
  241     / !
  0       / End of string
$HELLO    /DEFAULT TERMINATOR
Controversy over Computer Proof

Appel and Haken Proof (1976).

- Human part of the proof is over 1000 pages long and no one else has ever been able to verify it. Many typos were found.
- Computer portion of the proof is written in assembly language and no one else has programmed it.
- 1478 graphs had to be coded by hand.

Question: Are you convinced they have a proof?
Good ideas in the Appel-Haken Proof

Outline of Proof:
Assume G is a counterexample to the 4CT with a minimal number of vertices.

• Reducibility (human only).
• Unavoidability (computer assisted).
• Algorithm for finding a coloring in G
Good ideas in the Appel-Haken Proof

Outline: Assume G is a counterexample to the 4CT with a minimal number of vertices.

• Reducibility: AH give a finite list of 1478 configurations in graphs. Each one of these cannot appear in G because if it did, they gave rules to replace G by a smaller graph that would still be planar and require more than 4 colors.
Good ideas in the Appel-Haken Proof

- **Unavoidability**: Every minimal counterexample to the 4CT must contain one of the configurations on the list.

A configuration is a small neighborhood in a graph. AH prove they only need to look at the second neighbors of each vertex and they bound the number of neighbors in each configuration. There are only a finite number of such graphs.
Good ideas in the Appel-Haken Proof

Outline: Assume G is a counterexample to the 4CT with a minimal number of vertices.

• **Reducibility:** AH give a finite list of 1478 configurations in graphs which cannot appear in G.

• **Unavoidability:** G must contain one of the configurations on the list.

Question: What can you conclude about G?
Good ideas in the Appel-Haken Proof

- **Reducibility** (human only).
- **Unavoidability** (computer assisted).

Together imply there always exists a 4 coloring of any planar graph. But how do we find one?

- **Algorithm** for finding a 4-coloring in $G$. Guaranteed to succeed if a 4-coloring exists.
History

1996: “A New Proof of the Four Color Theorem” Published by Robertson, Sanders, Seymour, and Thomas based on the same outline.

• Human part of the proof is about 20 pages.
• Computer portion of the proof was originally written in C and several other people have independently programmed it.
• No graphs had to be coded by hand.
• Only 633 configurations used.

Question: Are you convinced they have a proof?
Some of the 633 Configurations
History

1996: “A New Proof of the Four Color Theorem” Published by Robertson, Sanders, Seymour, and Thomas based on the same outline.

• Algorithm: RSST also give an algorithm to find a 4-coloring of a planar graph that takes about \( n^2 \) seconds on a graph with \( n \) vertices.
Kepler’s Conjecture

Astronomers were wondering:

What is the best way to pack cannon balls in space so they are as close as possible?
Hexagonal Close Packing

http://upload.wikimedia.org/wikipedia/commons/8/8e/Close-packed_spheres.jpg
Kepler’s Conjecture

What is the best way to pack cannon balls in space so they are as close as possible?

Conjecture: The portion of space filled by cannonballs in the densest possible packing is given by the hexagonal close packing and has density

\[
\frac{\pi}{\sqrt{18}} \approx 0.7404804898.
\]
History of Hales Proof

• 1953: Toth showed that the problem could be reduced to a finite check of about 5000 cases.
• 1992: Thomas Hales and Samuel Ferguson began using linear programming to check the density of these cases.
• 1996: Hales announced the proof was complete.
• 2005: Hales’ paper was published after being reviewed by a committee of 12 referees who said they were 99% certain it was correct.
A halting problem

Problem: Find all graph types corresponding with rank 5 starred strong tableaux under cloning.

Human part of the proof is 50 pages long. It is ready to publish. Computer part has been running since January, but hasn’t finished.

Questions: When should we submit it for publication? Do you think it will be controversial?
Philosophical Question

What is the value of a computer proof?

• We get a new result which we can build on!
• We learn one more method of using computers to prove theorem.
• Every computer proof with no human proof contains a miracle which makes it computable!
Summary

Computers can be very helpful proving theorems about...

• Algebraic identities.
• Finite calculations.
• Halting problems

And what else?
• Origami: Can you fold this? See “Geometric Folding Algorithms” by Demaine and O’Rourke.


• Game Theory: Does this game have a winning strategy? See history of Connect Four in Wikipedia.