Rotations, reflections, and rearrangements

UW Math Hour

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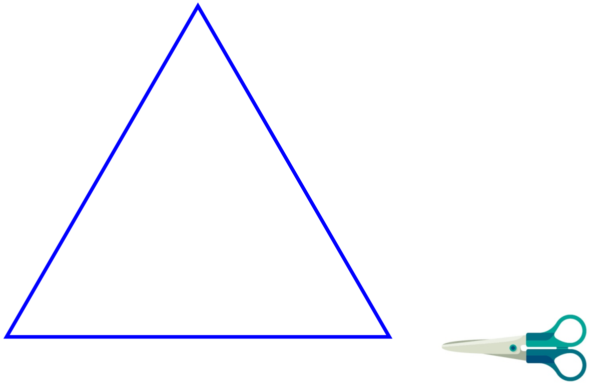
Triangles



Definition

Imagine that you have cut the triangle out of this piece of paper. A **symmetry** is an operation you can perform on the triangle so that it fits exactly back into the hole it was cut from.

Triangles

- How many symmetries does the equilateral triangle have? (Hint: use your triangle and perform rigid motions of it.) Come up with a description of the symmetries.

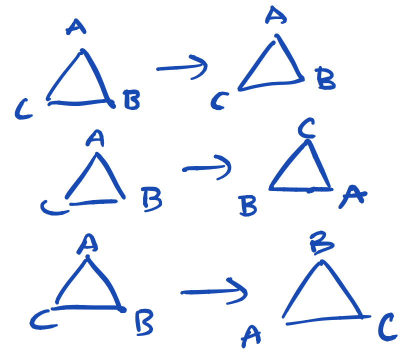
- Prove that you have found all symmetries of the triangle.

Triangles

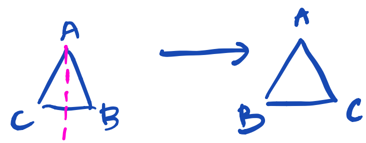
- What are the symmetries of the triangle?

Rotational symmetries:

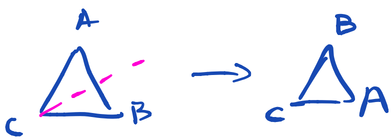
- do nothing = rotate 3 times : "N"
- rotate clockwise : "Rc"
- rotate clockwise 2 times [rotate counterclockwise] : "Rcc"



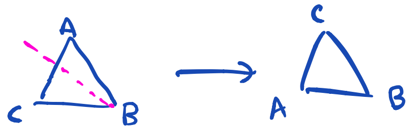
Reflection symmetries:

- 

"F_1"

- 

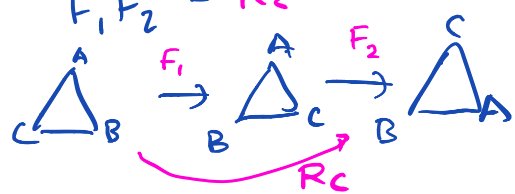
"F_2"

- 

"F_3"

Q. what if we do F_1 then F_2?

$F_1 F_2 = R_c$



Triangles

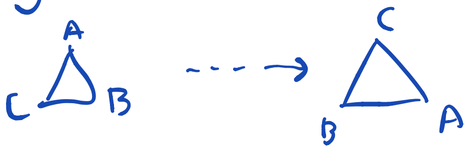
why are there only 6?

- How can we prove that we have found all of the symmetries?

· $3! = 6$

$3! = 3 \cdot 2 \cdot 1 =$ # of ways to rearrange 3 things

Any symmetry of the triangle is determined by the arrangement of the vertices:

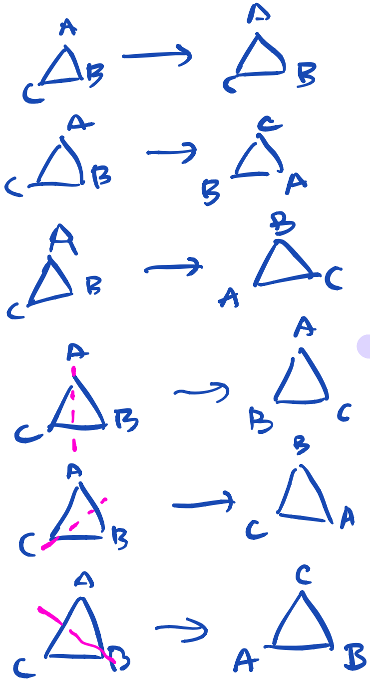


there are 6 ways to arrange the vertices, so there are only 6 symmetries.

Triangles

do one symmetries, and then another

- What happens if you **compose** two symmetries?



	N	R_c	R_{cc}	F_1	F_2	F_3
N	N	R_c	R_{cc}	F_1	F_2	F_3
R_c	R_c	R_{cc}	N	F_3	F_1	F_2
R_{cc}	R_{cc}	N	R_c	F_2	F_3	F_1
F_1	F_1	F_2	F_3	N	R_c	R_{cc}
F_2	F_2	F_3	F_1	R_{cc}	N	R_c
F_3	F_3	F_1	F_2	R_c	R_{cc}	N

observations.

- smaller grids look similar
- not symmetric $R_c \cdot F_1 \neq F_1 \cdot R_c$
- R's + N's : still get R's, N's
  BUT F's + F's get R's + N's
- each entry appears exactly once in each row and column

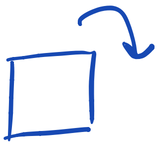
- What observations can we make about this table?

Squares

- Let's *step it up*: what about squares? How many symmetries are there?

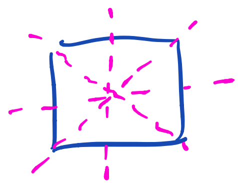
8 symmetries:

4 Rotational



rotate $0^\circ, 90^\circ, 180^\circ, 270^\circ$

4 Reflectional



Q. How to prove that these are all?

Squares

- What observations can we make about this table?

Other shapes!

For fun, you can take some other symmetric looking objects in your home (hexagons! pyramids! cubes!) and explore the tables you can make of their symmetry operations.

Rearrangements

Start with 5 tiles numbered $1, 2, 3, 4, 5$ and lay them at random along the squares of a 5×1 rectangle:

3	2	4	1	5

any order!

We will call the *standard configuration* the most natural one:

1	2	3	4	5

We'll encounter a few different puzzles, with the goal to move tiles in certain ways and turn any configuration into the standard one.

Question 1

5 choices 1,2,3,4,5	4 choices	3 choices	2	1

- How many configurations of tiles are there? What if there were n tiles instead of 5?

5 tiles: $5 \times 4 \times 3 \times 2 \times 1 = 5!$ configurations

$= 120$

n tiles: $n(n-1)(n-2)\cdots(2)(1) = n!$ configurations

Question 2

3	2	4	5	1

- If you are only allowed to swap two tiles at a time, can you always get the tiles into the standard configuration? What is the minimal number of moves needed?

- First, swap 1 with whatever tile is in the first position (if necessary)

1	2	4	5	3

- swap 2 w/ second pos.

1	2	4	5	3

- swap 3 w/ third pos.

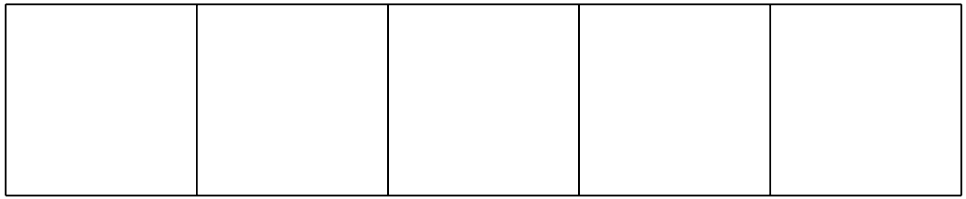
1	2	3	5	4

- swap 4 w/ fourth pos.

1	2	3	4	5

1, 2, 3, 4 are in correct pos. so 5 must also be!

Question 3



- If you are only allowed to swap two tiles at a time, but one of them **must be the tile in the first position** can you always get the tiles into the standard configuration?

Question 4

2	3	4	1	5

- If you are only allowed to pick three tiles and cyclically rotate them to the right (so, if you picked the tiles in spots 2, 4, and 5, the tile in 2 would go to 4, 4 would go to 5, and 5 would go to 2), can you always get the tiles into the standard configuration?

2	5	4	3	1

can you get **any** config. to the standard one if you can only do 3-cyclic rotations?

ex.

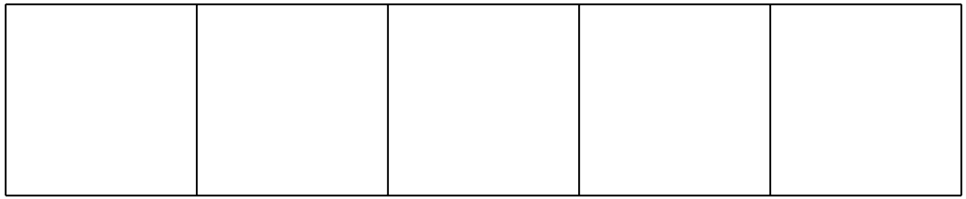
1	2	3	5	4

: Why no?

Idea: rotating 3 tiles, it will always affect another tile

Ans. No! not possible

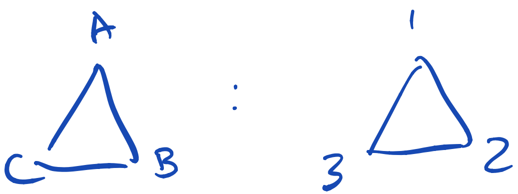
Question 5



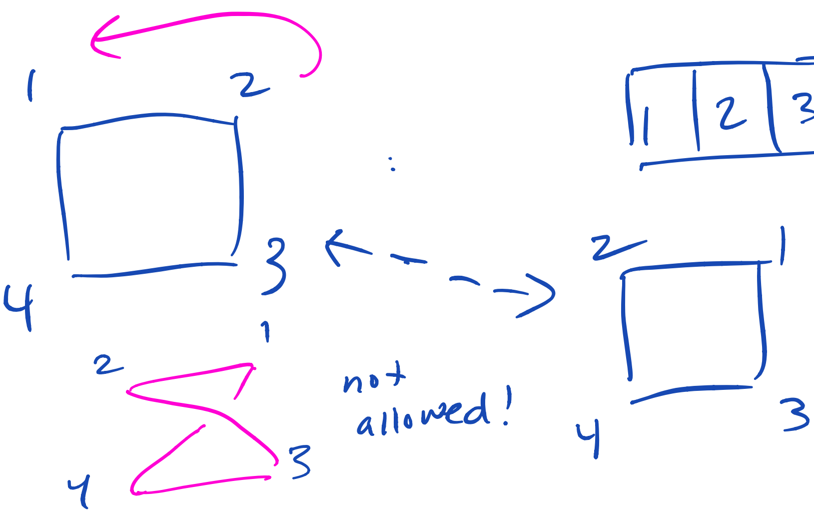
- If you are only allowed to swap the first two tiles or cyclically rotate any three tiles to the right, can you always get the tiles into the standard configuration?

Rearrangements and Symmetries

- Can you relate the rigid motions of the triangle to rearrangements of just three tiles? Can you relate the rigid motions of the square to rearrangements of four tiles?



symmetries corresponded exactly to rearrangements of vertices (= 3 tiles)



1	2	3	4

$\xrightarrow[\text{2 tiles}]{\text{swap}}$

2	1	3	4

*can't get this from a motion of the square!

· 1 and 3 are opposite and they cannot become adjacent.

Rearrangements and Symmetries

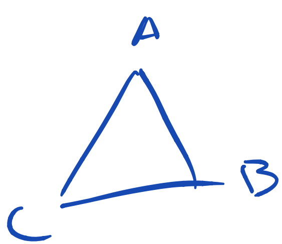
- If you consider the symmetries of the shapes, and then all of ways that you could rearrange the tiles, what do these sets have in common?

Groups

Both of these are examples of **groups.**

A **group** is a set of objects G with an operation $\star$ satisfying three properties:

- There is a **do nothing** object. "identity"
- Each object has an **undo** object. "inverse"
- The operation is **associative**: $a \star (b \star c) = (a \star b) \star c$.



do nothing ✓

undo: $R_c \longrightarrow$ undo: R_{cc}

Group Examples

- The **dihedral group** is the set of symmetries of an n-gon.
- The **symmetric group** is the set of rearrangements of n tiles.
- The **integers** are a group, where the operation is *addition.*
  - Do nothing?
  - Undo?
  - Associativity?

$\{\ldots, -3, -2, -1, 0, 1, 2, 3 \ldots\}$

what can we add to...

- do nothing? $+0$
- undo? If we add $+3$, add -3

undo: $+n$: $-n$

Another Puzzle

- The 15 puzzle is a puzzle on a 4×4 grid that has 15 numbered squares and one empty square. You are allowed to move the tiles only by sliding squares into the empty square. Can you get between the following two configurations by sliding the tiles into the empty square?

* related to problem about rotating tiles in groups of 3 *

1	2	3	4
5	6	7	8
9	10	11	12
13	15	14	

$\Rightarrow$

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

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More Group Things!

- This puzzle was recently featured on **Numberphile**! I'll put the link here:
  https://www.numberphile.com/videos/15-to-1-puzzle
- Groups are used in real life in a different ways! They are used in...
  - RSA Encryption (a way to send data securely on the internet)
  - Advanced chemistry and physics (computing where very small particles are likely to be at any given time, and how that's related to molecular properties)
  - More! Mathematicians even study (unsolved!) questions about groups.


THANK YOU